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An advection-diffusion equation within a nonlinear degenerate diffusion in a diffuse interface framework A liquid-vapour flows with phase transition in an heat exchanger

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joint work with C. Galusinski, B. Grec and Y. Penel





Outline

- 1. Introduction
- 2. Steady-state model
- 3. Simplified Model
- 4. Full Model
- 5. Summary

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1. Introduction

- 1.1 Context
- 1.2 The Low Mach Hypothesis
- 1.3 A Low Mach number model for a heat exchanger
- 1.4 The LMNC model WITHOUT Thermal Diffusion
- 1.5 Diphasic equation of state with phase transition
- 1.6 The Final model

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Pressurized Water Reactor



Pressurized Water Reactor



Core of a Pressurized Water Reactor



A heat exchanger



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Core at Pressurized Water Reactor

Nominal regime

- Inlet velocity: $|\mathbf{u}_e| \approx 5 \,\mathrm{m\,s^{-1}}$
- At $p_0 = 155$ bar and $T = 300\,^\circ$ C: speed of sound $c_\ell^* \simeq 1 \times 10^3$ m s⁻¹
- Mach number (measure of compressibility) ${
 m M}={|{f u}_e|\over c_*^*}\simeq 5 imes 10^{-3}\ll 1$



This hypothesis also applies to:

- Incidental Regime
- Certain accidental scenarios, such as a LOFA (Loss of Flow Accident)¹ even if phase change occurs

¹Except for very rapid depressurization scenarios like a LOCA (Loss of Coolant Accident) induced by a coolant pump trip event

Which model?

A model with acoustics $M \geq 1$ and heat transfers

→ Compressible Navier-Stokes/Euler system.

ullet Acoustics negligible (no shock waves) ${
m M}\ll 1$

• High heat transfers: $abla \cdot \mathbf{u}
eq 0$

~ An asymptotic low Mach number model

A model without acoustics $\mathrm{M}=0$ and $abla\cdot\mathbf{u}=0$

~ Incompressible model

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→ Compressible Navier-Stokes/Euler system.

- \bullet Acoustics negligible (no shock waves) $M\ll 1$
- High heat transfers: $\nabla \cdot \mathbf{u} \neq 0$

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From Compressible Navier-Stokes-Fourier System to the LMNC model

Compressible Navier-Stokes-Fourier system 🛶 a Low Mach Number Model

 $\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho \mathbf{g} + \nabla \cdot \sigma(\mathbf{u})\\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi + \nabla \cdot (\omega \nabla T) + \sigma(\mathbf{u}) : \nabla \mathbf{u} + \partial_t p + \mathbf{u} \cdot \nabla p \end{cases}$

In Low Mach Number Regime we have $p(t,\mathbf{x})=p_*+\mathrm{M}^2ar{p}(t,\mathbf{x})$.

Unknowns

- $\bullet \ (t, \mathbf{x}) \mapsto \mathbf{u} \text{ velocity field}$
- $\bullet \ (t,{\bf x})\mapsto h \ {\rm enthalpy}$
- $(t, \mathbf{x}) \mapsto p$ pressure

Given

- $(t,\mathbf{x})\mapsto\Phi\geq 0$ power density modelling the heating
- $\bullet \ \omega$ heat conductivity
- ${\boldsymbol{g}}$ gravity field, $\sigma({\mathbf{u}})$ viscous effects
- $p_{st} > 0$ thermodynamic pressure (consta
- EoS: $(h,p_{-}) \mapsto \rho$ (density) and $(h,p_{-}) \mapsto T$ (temperature)

From Compressible Navier-Stokes-Fourier System to the LMNC model

Compressible Navier-Stokes-Fourier system ~ a Low Mach Number Model

 $\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p} = \rho \mathbf{g} + \nabla \cdot \sigma(\mathbf{u})\\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi + \nabla \cdot (\omega \nabla T) + \sigma(\mathbf{u}) : \nabla \mathbf{u} + \partial_t \mathbf{p}_* + \mathbf{u} \cdot \nabla \mathbf{p}_*\end{cases}$

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- $(t, \mathbf{x}) \mapsto \mathbf{u}$ velocity field
- $\bullet \ (t,{\bf x})\mapsto h \ {\rm enthalpy}$
- $(t, \mathbf{x}) \mapsto \bar{p}$ perturbational pressure

• $(t, \mathbf{x}) \mapsto \Phi \ge 0$ power density modelling the heating

- ω heat conductivity
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- $(t,\mathbf{x})\mapsto\Phi\geq 0$ power density modelling the heating
- ω heat conductivity
- ${m g}$ gravity field, $\sigma({f u})$ viscous effects
- $p_* > 0$ thermodynamic pressure (constant)
- EoS: $(h,p_*) \mapsto \rho$ (density) and $(h,p_*) \mapsto T$ (temperature)

The model

By neglecting the viscous terms and the dependency on the constant pressure p_* , the model becomes

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \bar{p} = \rho \mathbf{g}\\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi + \nabla \cdot (\omega \nabla T) \end{cases}$$

Unknowns

- $\bullet \ (t, \mathbf{x}) \mapsto \mathbf{u} \text{ velocity field}$
- $(t, \mathbf{x}) \mapsto h$ enthalpy
- $(t, \mathbf{x}) \mapsto \bar{p}(t, \mathbf{x})$ perturbational pressure

Given

- $\Phi > 0$ constant power density
- ullet $egin{array}{c} g & {
 m gravity field} \end{array}$
- ω heat conductivity

Closure

Diphasic equation of state: $h \mapsto T(h)$ (temperature) and $h \mapsto \rho(h)$ (specific density)

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Summary

- Analytical Solutions (1D)
 - Steady-state for every choice of EoS for pure phases
 - Transitory solutions for Noble-Abel Stiffened-Gas (NASG) for pure phases
- Numerical Schemes
 - 1D non-cons. formulation: MOC, INTMOC (exponential), WB schemes
 - 3D non-cons. formulation: prediction-projection method
- EoS with Phase Transition at Saturation
 - SG, NASG for pure phases
 - $\bullet~\text{IAPWS} \rightsquigarrow \text{incomplete SG exact on the boundaries of the saturation}$
 - Cubic (Van der Waals, Berthelot, Clausius...) with Maxwell construction

- A Relaxation Model (4-LMNC model)
 - EDP + BGK term
 - EoS (Chemical Potential Disequilibrium)
 - Formal Convergence
 - WB and AP Schemes

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G.F., B. Grec, Y. Penel, ESAIM: M2AN 2021

The conservative model ~> a non-conservative formulation

Conservative formulation

 $\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \bar{p} = \rho \boldsymbol{g}\\ \partial_t (\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \Phi \end{cases}$

$$f = \tau \stackrel{\text{def}}{=} 1/\varrho$$

Non-conservative formulation

 $\begin{cases} \partial_t h + \mathbf{u} \cdot \nabla h = \Phi(t, \mathbf{y}) \tau(h) \\ \nabla \cdot \mathbf{u} = \Phi(t, \mathbf{y}) \tau'(h) \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \tau(h) \nabla \bar{p} = \mathbf{g} \end{cases}$

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Conservative formulation

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1D Conservative formulation

 $\begin{cases} \partial_t \rho + \partial_y (\rho v) = 0\\ \partial_t (\rho h) + \partial_y (\rho h v) = \Phi \end{cases}$ and $\partial_y \bar{p} = -\rho g - \partial_t (\rho v) - \partial_y (\rho v^2)$

Non-conservative formulation

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1D Non-conservative formulation $\begin{cases} \partial_t h + v \partial_y h = \Phi \tau(h) \\ \partial_y v = \Phi \tau'(h) \end{cases}$ and $\tau(h)\partial_y \bar{p} = -g - \partial_t v - v \partial_y v$

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Schemes

We employ a standard projection method based on a non-conservative formulation on a staggered grid (\simeq for Navier-Stokes):

I First, we solve the transport equation for enthalpy.

$$\partial_t h + \mathbf{u} \cdot \nabla h = \Phi(t, \mathbf{y}) \tau(h)$$

② Next, the other equations are solved using a pressure-correction method.

$$\begin{cases} \nabla \cdot \mathbf{u} = \Phi(t, \mathbf{y}) \tau'(h) \\\\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \tau(h) \nabla \bar{p} = \mathbf{g} \end{cases}$$

This involves two substeps per time step (repeated until the divergence reaches the desired value):

- Velocity prediction considering the pressure explicit in the first substep.
- Pressure correction in the second substep by projecting the predicted velocity onto the space of a "divergence-fixed" field.

Next Step: What happens with thermal diffusion?

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• Diffuse Interface Framework:

Pure phase κ ∈ { ℓ, _θ } is described by a given (complete) EoS ~→

 $(h,p)\mapsto T_{\kappa}(h,p)$

• Mixture: at saturation (same pressure p, temperature T, chemical potential μ)

 $\mu_{\mathbb{I}}(T,p)=\mu_{\mathfrak{F}}(T,p) \rightsquigarrow p \mapsto T^{\mathrm{sat}}(p)$

temperature at saturation

• Transition pure phase/mixture: $h_\kappa^{
m sat}(p) \stackrel{
m def}{=} h_\kappa(T^{
m sat}(p),p)$ the enthalpy of the phase κ at saturation

At pressure p, the fluid is

- ullet in the liquid phase if $h\leq h_{\mathbb{I}}^{ ext{sat}}(p)$
- ullet a mixture at saturation if $h_{\ell}^{ ext{sat}}(p) < h < h_{9}^{ ext{sat}}(p)$

• in the vapor phase if $h \ge h_g^{\rm sat}(p)$

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the (compressible) fluid can exist in liquid (l) or vapor ($_{\vartheta})$ phase or as a mixture of both

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• Diffuse Interface Framework:

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 Mixture: at saturation (same pressure p, temperature T, chemical potential μ)

$$\mu_{\mathbb{I}}(T,p)=\mu_{\mathrm{G}}(T,p) \leadsto p \mapsto T^{\mathrm{sat}}(p)$$

temperature at saturation

• Transition pure phase/mixture: $h_{\kappa}^{\text{sat}}(p) \stackrel{\text{def}}{=} h_{\kappa}(T^{\text{sat}}(p), p)$ the enthalpy of the phase κ at saturation

At pressure p, the fluid is

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- \bullet a mixture at saturation if $h_{\ell}^{\rm sat}(p) < h < h_{\mathfrak{F}}^{\rm sat}(p)$
- in the vapor phase if $h \geq h_{\Re}^{\scriptscriptstyle{
 m sol}}(p)$

Temperature with Phase Transition at Constant Pressure

• T is piecewise defined w.r.t. $h_{\kappa}^{\rm sat}(p)$

$$(h,p) \mapsto T(h,p) = \begin{cases} T_{\mathbb{l}}(h,p), & \text{if } h \leq h_{\mathbb{l}}^{\text{sal}}(p) \\ T^{\text{sal}}(p), & \text{if } h_{\mathbb{l}}^{\text{sal}}(p) < h < h_{\mathfrak{g}}^{\text{sal}}(p) \\ T_{\mathfrak{g}}(h,p), & \text{if } h \geq h_{\mathfrak{g}}^{\text{sal}}(p) \end{cases}$$

• In our EDP system, the **thermodynamic** pressure p_* is constant



Temperature with Phase Transition at Constant Pressure

• T is piecewise defined w.r.t. $h_{\kappa}^{\rm sal}(p)$

$$(h,p) \mapsto T(h,p) = \begin{cases} T_{\mathbb{I}}(h,p), & \text{if } h \leq h_{\mathbb{I}}^{\text{sal}}(p) \\ T^{\text{sal}}(p), & \text{if } h_{\mathbb{I}}^{\text{sal}}(p) < h < h_{\vartheta}^{\text{sal}}(p) \\ T_{\vartheta}(h,p), & \text{if } h \geq h_{\vartheta}^{\text{sal}}(p) \end{cases}$$

• In our EDP system, the **thermodynamic** pressure p_* is constant



Diffusion term

Temperature in the LMNC model

• LMNC model

$$\partial_t(\rho h) + \nabla \cdot (\rho h \mathbf{u}) = \left[\Phi(t, \mathbf{y}) + \nabla \cdot \left(\omega(h, p_*) \nabla T(h, p_*) \right) \right]$$

• Mixture at saturation and thermodynamic pressure p_* constant:



 $\omega(h)\nabla T(h) = \begin{cases} \lambda_{\mathbb{I}}\nabla h, & \text{if } h \le h_{\mathbb{I}}^{\text{sat}}, \\ 0, & \text{if } h_{\mathbb{I}}^{\text{sat}} < h < h_{\mathfrak{g}}^{\text{sat}}, \\ \lambda_{\mathfrak{g}}\nabla h, & \text{if } h \ge h_{\mathfrak{g}}^{\text{sat}}, \end{cases}$

$$\begin{array}{l} \lambda_{\kappa} \stackrel{\mathrm{def}}{=} \frac{\omega_{\kappa}}{c_{p,\kappa}} \\ c_{p,\kappa} \stackrel{\mathrm{def}}{=} \frac{\partial h}{\partial T} \Big|_{p} \text{ isobar heat capacity} \end{array}$$

Diffusion term

Temperature in the LMNC model

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The 1D model

9 v and h are solutions of

$$\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0 \\ \partial_t (\varrho h) + \partial_y (\varrho h v) = [\Phi + \partial_y (\lambda(h) \partial_y h)] \end{cases} \qquad \qquad h \mapsto \lambda(h) = \begin{cases} \lambda_{\ell} & \text{if } h \leq h_{\ell}^{\text{adt}} \\ 0 & \text{if } h_{\ell}^{\text{adt}} < h < h_{\mathfrak{g}}^{\text{adt}} \\ \lambda_{\mathfrak{g}} & \text{if } h \geq h_{\mathfrak{g}}^{\text{adt}} \end{cases}$$
$$h \mapsto \rho(h)$$

- Domain: $y \in \mathbb{R}^+$
- Boundary conditions:

• y=0: Dirichlet (injection) $v(t,0)=v_e>0$ and $h(t,0)=h_e< h_{\mathbb{Q}}^{ ext{salt}}$ constant,

- $y \to +\infty:$ asymptotic behavior $\lim_{y \to +\infty} \partial_y h(t,y) = \Phi/D_e$
- Constants: $\Phi > 0$, $D_e \stackrel{\text{def}}{=} v_e \varrho(h_e) > 0$
- Initial conditions: $h(0,y) = h_e$ liquid phase

2 Additionally, \bar{p} is a solution of

$$\partial_y \bar{p} = -\varrho g - \partial_t (\varrho v) - \partial_y (\varrho v^2)$$

2. Steady-state model

- 2.1 The 1D steady-state model
- 2.2 Solution with $\lambda_{\ell} = \lambda_m = \lambda_g = 0$ (without diffusion)
- 2.3 Solution with $\lambda_{\ell}, \lambda_{m}, \lambda_{g} > 0$ (strictly positive diffusion)
- 2.4 Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_g > 0$ (degenerate diffusion) 2.5 Link to the Stationary Stefan Problem
- 2.6 Conclusion on the Steady-State Model

2. Steady-state model

2.1 The 1D steady-state model

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- 2.3 Solution with $\lambda_{\ell}, \lambda_{m}, \lambda_{g} > 0$ (strictly positive diffusion)
- 2.4 Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_g > 0$ (degenerate diffusion
- 2.5 Link to the Stationary Stefan Problem
- 2.6 Conclusion on the Steady-State Mode

$$\begin{cases} \frac{\partial_y(\varrho v) = 0}{\partial_y(\varrho vh) = \left[\Phi + \partial_y\left(\lambda(h)\partial_y h\right)\right]} & y \in [0; +\infty) \\ \frac{\partial_y(\varrho vh) = \left[\Phi + \partial_y\left(\lambda(h)\partial_y h\right)\right]}{(\varrho v)(0) = \varrho(h_e)v_e \stackrel{\text{def}}{=} D_e > 0 \text{ constant} \\ \lim_{y \to +\infty} h'(y) = \frac{\Phi}{D_e} \end{cases} \\ \downarrow \\ \begin{cases} (\varrho v)(y) = D_e \ \forall y \in \mathbb{R}^+ & \rightsquigarrow v(y) = \frac{D_e}{\varrho(h(y))} \\ D_e \partial_y h = \left[\Phi + \partial_y\left(\lambda(h)\partial_y h\right)\right] & \text{ independent of } h \mapsto \varrho(h) \\ h(0) = h_e \\ \lim_{y \to +\infty} h'(y) = \frac{\Phi}{D_e} \end{cases}$$

EDO
$$D_e h'(y) - \left(\lambda(h)h'(y)\right)' = \Phi$$

- Constants: $\Phi > 0$, $D_e > 0$
- Domain: $[0, +\infty)$
- Boundary conditions:
 - Dirichlet (inlet) $h(0) = h_e < h_0^{\text{salt}}$
 - Asymptotic behavior $\lim_{y
 ightarrow\infty}h'(y)=rac{\Phi}{D_e}$

Diffusion		
$\lambda(h) = \begin{cases} \\ \end{cases}$	$egin{array}{l} \lambda_{\ell} \ \lambda_{m} \ \lambda_{artheta} \end{array}$	$\begin{split} \text{if } h &\leq h_{\ell}^{\text{\tiny Adl}} \\ \text{if } h_{\ell}^{\text{\tiny Adl}} &< h < h_{\vartheta}^{\text{\tiny Adl}} \\ \text{if } h &\geq h_{\vartheta}^{\text{\tiny Adl}} \end{split}$

We will consider three cases:

•
$$\lambda_{\ell} = \lambda_m = \lambda_g = 0$$
: model without diffusion

• $\lambda_{\ell}, \lambda_{m}, \lambda_{g} > 0$: model with strictly positive diffusion

• $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_{q} > 0$: model with a degenerate diffusion

Remark

Due to the discontinuities in λ , the ODE should be interpreted as follows:

 $D_e h' - (L \circ h)'' = \Phi$ with $L'(h) = \lambda(h)$

EDO

- $D_e h'(y) \left(\lambda(h)h'(y)\right)' = \Phi$
- Constants: $\Phi > 0$, $D_e > 0$
- Domain: $[0, +\infty)$
- Boundary conditions:
 - Dirichlet (inlet) $h(0) = h_e < h_{\mathbb{Q}}^{\mathrm{salt}}$
 - Asymptotic behavior $\lim_{y
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Diffusior	า	
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2. Steady-state model

2.1 The 1D steady-state model

2.2 Solution with $\lambda_{\ell} = \lambda_m = \lambda_g = 0$ (without diffusion)

- 2.3 Solution with $\lambda_{\ell}, \lambda_{m}, \lambda_{q} > 0$ (strictly positive diffusion)
- 2.4 Solution with $\lambda_{
 m m}=0$ and $\lambda_{
 m l},\lambda_{
 m g}>0$ (degenerate diffusion
- 2.5 Link to the Stationary Stefan Problem
- 2.6 Conclusion on the Steady-State Mode

Proposition 1

If $\lambda(h) \equiv 0$ the steady enthalpy solution is

$$h(y) = h_e + \frac{\Phi}{D_e}y$$

Proof.

$$\begin{cases} D_e h'(y) = \Phi \\ h(0) = h_e \\ \lim_{y \to +\infty} h'(y) = \frac{\Phi}{D_e} \end{cases} \implies h(y) = h_e + \frac{\Phi}{D_e} y$$



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Proposition 2

Without degeneracy of diffusion, the unique solution is continuous and can be written as

$$h(y) = \begin{cases} h_{\ell}(y) \stackrel{\text{def}}{=} C_{\ell,1} + \frac{\Phi}{D_e} y + C_{\ell,2} \exp\left(\frac{y}{\lambda_{\ell}/D_e}\right) & \text{if } y \leq y_{\ell}^{\text{sol}} \\ \\ h_m(y) \stackrel{\text{def}}{=} C_{m,1} + \frac{\Phi}{D_e} y + C_{m,2} \exp\left(\frac{y}{\lambda_{m}/D_e}\right) & \text{if } y_{\ell}^{\text{sol}} \leq y < y_{\mathcal{J}}^{\text{sol}} \\ \\ h_{\mathcal{J}}(y) \stackrel{\text{def}}{=} h_{\mathcal{J}}^{\text{sol}} + \frac{\Phi}{D_e} (y - y_{\mathcal{J}}^{\text{sol}}) & \text{if } y \geq y_{\mathcal{J}}^{\text{sol}} \end{cases}$$

where the 4 constants $C_{\kappa,1}$ and $C_{\kappa,2}$ depend on y_{ℓ}^{sat} and y_{g}^{sat} .

The boundaries $y_{\ell}^{\scriptscriptstyle a a \ell}$ and $y_{\scriptscriptstyle g}^{\scriptscriptstyle a a \ell}$ satisfies the continuity flux

$$\lambda_\ell h'_\ell(y_\ell^{\mathit{sal}}) = \lambda_{\scriptscriptstyle m} h'_{\scriptscriptstyle m}(y_\ell^{\mathit{sal}}) \quad \textit{and} \quad \lambda_{\scriptscriptstyle m} h'_{\scriptscriptstyle m}(y_g^{\mathit{sal}}) = \lambda_g h'_g(y_g^{\mathit{sal}}).$$

Proof – Solution in Each Region.

• Given $\Phi > 0$ and $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In each region, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_b/D_e}\right)$

- The boundary conditions give two relations:
 - Liquid region: $h_{l}(0)=h_{e} \rightsquigarrow C_{l,2}=h_{e}-C_{l,1}$
 - Vapor region: $\lim_{y o\infty}h_q'(y)=rac{2}{D_q} weeksymbol{\sim} C_{q,2}=0$.
- We need to compute $C_{\ell,1}$, $C_{m,1}$, $C_{m,2}$, $C_{g,1}$ and the transition points y_{ℓ}^{soft} , y_{g}^{soft} .

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 - Liquid region: $h_l(0) = h_e \sim O_{1,2} = h_e O_{1,1}$
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 - Given $\Phi > 0$ and $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

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- The boundary conditions give two relations:
 - Liquid region: $h_{\mathbb{l}}(0) = h_e \rightsquigarrow C_{\mathbb{l},2} = h_e C_{\mathbb{l},1}$
 - Vapor region: $\lim_{y\to\infty}h'_{\vartheta}(y)=\frac{\Phi}{D_e} \rightsquigarrow C_{\vartheta,2}=0$

• We need to compute $C_{\ell,1}$, $C_{m,1}$, $C_{m,2}$, $C_{g,1}$ and the transition points $y_\ell^{_{hal}}$, $y_g^{_{hal}}$.

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- Proof Solution in Each Region.
 - Given $\Phi > 0$ and $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In each region, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\parallel}/D_e}\right)$.

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• Liquid region:
$$h_{\ell}(0)=h_e \rightsquigarrow C_{\ell,2}=h_e-C_{\ell,2}$$

• Vapor region: $\lim_{y\to\infty} h'_{\theta}(y) = \frac{\Phi}{D_e} \rightsquigarrow C_{\theta,2} = 0$

• We need to compute $C_{\ell,1}$, $C_{m,1}$, $C_{m,2}$, $C_{g,1}$ and the transition points y_{ℓ}^{sat} , y_{g}^{sat} .

Proof – Transition Points.

- Continuity at $y_{\ell}^{\text{\tiny halt}}$ and $y_{g}^{\text{\tiny halt}}$:
 - Liquid region: $h_{l}(y_{l}^{\text{sat}}) = h_{l}^{\text{sat}} \rightsquigarrow C_{l,1}$ function of y_{l}^{sat} ;
 - Mixture region: $h_m(y_{\emptyset}^{\text{sat}}) = h_{\emptyset}^{\text{sat}}$, $h_m(y_q^{\text{sat}}) = h_q^{\text{sat}} \rightsquigarrow C_{m,1}, C_{m,2}$ functions of $y_{\emptyset}^{\text{sat}}$ and y_q^{sat}
 - Vapor region: $h_g(y_g^{ ext{sol}}) = h_g^{ ext{sol}} \rightsquigarrow C_{g,1}$ function of $y_g^{ ext{sol}}$

• The positions of the transition points, y_{l}^{Aat} and y_{g}^{Aat} , are **implicitly** determined by the continuity conditions of flux:

$$\lambda_{\ell} h'_{\ell}(y^{\scriptscriptstyle{\mathrm{adl}}}_{\ell}) = \lambda_{\scriptscriptstyle{\mathrm{m}}} h'_{\scriptscriptstyle{\mathrm{m}}}(y^{\scriptscriptstyle{\mathrm{adl}}}_{\ell}) \qquad ext{and} \qquad \lambda_{\scriptscriptstyle{\mathrm{m}}} h'_{\scriptscriptstyle{\mathrm{m}}}(y^{\scriptscriptstyle{\mathrm{adl}}}_{\mathfrak{g}}) = \lambda_{\mathfrak{g}} h'_{\mathfrak{g}}(y^{\scriptscriptstyle{\mathrm{adl}}}_{\mathfrak{g}})$$

Proof – Transition Points.

- Continuity at y_{l}^{sat} and y_{g}^{sat} :
 - Liquid region: $h_{\ell}(y_{\ell}^{\text{sal}}) = h_{\ell}^{\text{sal}} \rightsquigarrow C_{\ell,1}$ function of y_{ℓ}^{sal} ;
 - Mixture region: $h_m(y_{\emptyset}^{\text{sat}}) = h_{\emptyset}^{\text{sat}}$, $h_m(y_{g}^{\text{sat}}) = h_{g}^{\text{sat}} \rightsquigarrow C_{m,1}, C_{m,2}$ functions of $y_{\emptyset}^{\text{sat}}$ and y_{g}^{sat}
 - Vapor region: $h_g(y_q^{ ext{sat}}) = h_g^{ ext{sat}} woheadrightarrow C_{g,1}$ function of $y_q^{ ext{sat}}$

• The positions of the transition points, y_{l}^{sal} and y_{g}^{sal} , are **implicitly** determined by the continuity conditions of flux:

$$\lambda_{\ell} h'_{\ell}(y^{\scriptscriptstyle{ extsf{act}}}_{
m l}) = \lambda_{
m m} h'_{
m m}(y^{\scriptscriptstyle{ extsf{act}}}_{
m l}) \qquad ext{and} \qquad \lambda_{
m m} h'_{
m m}(y^{\scriptscriptstyle{ extsf{act}}}_{
m g}) = \lambda_{
m g} h'_{
m g}(y^{\scriptscriptstyle{ extsf{act}}}_{
m g})$$

Proof – Transition Points.

- Continuity at y_{l}^{sat} and y_{g}^{sat} :
 - Liquid region: $h_{\ell}(y_{\ell}^{\text{sal}}) = h_{\ell}^{\text{sal}} \rightsquigarrow C_{\ell,1}$ function of y_{ℓ}^{sal} ;
 - Mixture region: $h_m(y_l^{\text{sat}}) = h_l^{\text{sat}}$, $h_m(y_q^{\text{sat}}) = h_q^{\text{sat}} \rightsquigarrow C_{m,1}, C_{m,2}$ functions of y_l^{sat} and y_q^{sat} ;
 - Vapor region: $h_g(y_a^{
 m Aat}) = h_a^{
 m Aat} \rightsquigarrow C_{g,1}$ function of $y_a^{
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• The positions of the transition points, y_{l}^{Aut} and y_{g}^{Aut} , are **implicitly** determined by the continuity conditions of flux:

$$\lambda_{\ell} h'_{\ell}(y^{\scriptscriptstyle{\mathrm{sal}}}_{\ell}) = \lambda_{\scriptscriptstyle{\mathrm{m}}} h'_{\scriptscriptstyle{\mathrm{m}}}(y^{\scriptscriptstyle{\mathrm{sal}}}_{\ell}) \qquad ext{and} \qquad \lambda_{\scriptscriptstyle{\mathrm{m}}} h'_{\scriptscriptstyle{\mathrm{m}}}(y^{\scriptscriptstyle{\mathrm{sal}}}_{\vartheta}) = \lambda_{\scriptscriptstyle{\vartheta}} h'_{\vartheta}(y^{\scriptscriptstyle{\mathrm{sal}}}_{\vartheta})$$

Proof – Transition Points.

- Continuity at y_{l}^{sat} and y_{g}^{sat} :
 - Liquid region: $h_{\ell}(y_{\ell}^{\text{sal}}) = h_{\ell}^{\text{sal}} \rightsquigarrow C_{\ell,1}$ function of y_{ℓ}^{sal} ;
 - Mixture region: $h_m(y_l^{\text{soft}}) = h_l^{\text{soft}}$, $h_m(y_q^{\text{soft}}) = h_q^{\text{soft}} \rightsquigarrow C_{m,1}, C_{m,2}$ functions of y_l^{soft} and y_q^{soft} ;

• Vapor region:
$$h_g(y_g^{\text{solt}}) = h_g^{\text{solt}} \rightsquigarrow C_{g,1}$$
 function of y_g^{solt} .

• The positions of the transition points, y_{l}^{sat} and y_{g}^{sat} , are **implicitly** determined by the continuity conditions of flux:

$$\lambda_{\ell} h'_{\ell}(y^{\scriptscriptstyle{
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m Aal}}_{\ell}) \qquad {
m and} \qquad \lambda_{\scriptscriptstyle{
m m}} h'_{\scriptscriptstyle{
m m}}(y^{\scriptscriptstyle{
m Aal}}_{
m g}) = \lambda_{
m g} h'_{
m g}(y^{\scriptscriptstyle{
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Proof – Transition Points.

- Continuity at $y_{l}^{\scriptscriptstyle{\mathrm{sat}}}$ and $y_{\mathrm{g}}^{\scriptscriptstyle{\mathrm{sat}}}$:
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 - Mixture region: $h_m(y_{l}^{\text{salt}}) = h_{l}^{\text{salt}}$, $h_m(y_{g}^{\text{salt}}) = h_{g}^{\text{salt}} \rightsquigarrow C_{m,1}, C_{m,2}$ functions of y_{l}^{salt} and y_{g}^{salt} ;
 - Vapor region: $h_g(y_g^{\mathrm{sat}}) = h_g^{\mathrm{sat}} \rightsquigarrow C_{g,1}$ function of y_g^{sat} .
- The positions of the transition points, y_{l}^{sat} and y_{g}^{sat} , are **implicitly** determined by the continuity conditions of flux:

$$\lambda_{\mathbb{I}} h'_{\mathbb{I}}(y^{\text{sat}}_{\mathbb{I}}) = \lambda_{\mathbb{m}} h'_{\mathbb{m}}(y^{\text{sat}}_{\mathbb{I}}) \qquad \text{and} \qquad \lambda_{\mathbb{m}} h'_{\mathbb{m}}(y^{\text{sat}}_{\mathfrak{g}}) = \lambda_{\mathfrak{g}} h'_{\mathfrak{g}}(y^{\text{sat}}_{\mathfrak{g}})$$

Let's take a look at the graphs to observe the limit as $\lambda_m o 0$.

---- Jupyter Notebook



Let's take a look at the graphs to observe the limit as $\lambda_m \to 0$.

---- Jupyter Notebook



2. Steady-state model

- 2.4 Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_g > 0$ (degenerate diffusion) 2.5 Link to the Stationary Stefan Problem

Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_{g} > 0$ (degenerate diffusion)

Before delving into the analytical solution, let's take a look at the graphs \rightsquigarrow Jupyter Notebook

If $\lambda_m = 0$, we have two distinct cases:

- Liquid/mixture/gas
- Liquid/gas (the mixture is not permitted)

Remark

Due to the jump in h, the ODE $D_e h'(y) - (\lambda(h)h'(y))' = \Phi$ should be interpreted as follows:

$$D_e h' - (L \circ h)'' = \Phi \qquad L(h) \stackrel{\text{def}}{=} \begin{cases} \lambda_{\hat{l}} (h - h_{\hat{l}}^{\text{sal}}) & \text{if } h \leq h_{\hat{l}}^{\text{sal}} \\ 0 & \text{if } h_{\hat{l}}^{\text{sal}} < h < h_{\hat{g}}^{\text{sal}} \\ \lambda_{\hat{g}} (h - h_{\hat{g}}^{\text{sal}}) & \text{if } h \geq h_{\hat{g}}^{\text{sal}} \end{cases} \quad \text{(thus, } L'(h) = \lambda(h))$$

Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_{g} > 0$ (degenerate diffusion)

Before delving into the analytical solution, let's take a look at the graphs --- Jupyter Notebook

- If $\lambda_{m}=0$, we have two distinct cases:
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Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_g > 0$ (degenerate diffusion)

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Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_g > 0$ (degenerate diffusion)- I: Liquid/mixture/gas

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} < h_{\vartheta}^{\scriptscriptstyle \mathrm{aat}} - h_{\vartheta}^{\scriptscriptstyle \mathrm{aat}}$$

Summary of the observations:

- Mixture presence
- Position y_{l}^{sat} implicitly defined by $h_{l}(y_{l}^{\text{sat}}) = h_{l}^{\text{sat}}$ and we have $(h_{l})'(y_{l}^{\text{sat}}) = 0$
- Position y_g^{sat} computed w.r.t. y_l^{sat} by $y_g^{\text{sat}} = y_l^{\text{sat}} + rac{D_e}{\Phi}(h_g^{\text{sat}} h_l^{\text{sat}}) rac{\lambda_g}{D_e}$
- Gas diffusion reduces mixture region for steady solution $(y_q^{
 m sat}-y_\ell^{
 m sat})=(x_q^{
 m sat}-x_\ell^{
 m sat})-rac{\lambda_q}{D_r}$
- Jump occurs within the mixture region and

$$\llbracket h
rbracket(y^{ ext{sat}}_{rak{g}}) = h^{ ext{sat}}_{rak{g}} - h_{ ext{m}}(y^{ ext{sat}}_{rak{g}}) = rac{\lambda_{rak{g}}}{D_e} rac{\Phi}{D_e} \quad \left[< h^{ ext{sat}}_{rak{g}} - h^{ ext{sat}}_{rak{g}}
ight]$$

Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_g > 0$ (degenerate diffusion)- I: Liquid/mixture/gas

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} < h_{\vartheta}^{\scriptscriptstyle \mathrm{aat}} - h_{\varrho}^{\scriptscriptstyle \mathrm{aat}}$$

Summary of the observations:

- Mixture presence
- Position y_{ℓ}^{salt} implicitly defined by $h_{\ell}(y_{\ell}^{\text{salt}}) = h_{\ell}^{\text{salt}}$ and we have $(h_{\ell})'(y_{\ell}^{\text{salt}}) = 0$
- Position y_{g}^{solt} computed w.r.t. y_{l}^{solt} by $y_{g}^{\text{solt}} = y_{l}^{\text{solt}} + \frac{D_{e}}{\Phi}(h_{g}^{\text{solt}} h_{l}^{\text{solt}}) \frac{\lambda_{g}}{D_{e}}$
- Gas diffusion reduces mixture region for steady solution $(y_q^{\text{saft}} y_{\emptyset}^{\text{saft}}) = (x_q^{\text{saft}} x_{\emptyset}^{\text{saft}}) rac{\lambda_q}{D_p}$
- Jump occurs within the mixture region and

$$\llbracket h
rbracket(y^{ ext{sat}}_{ ext{g}}) = h^{ ext{sat}}_{ ext{g}} - h_{ ext{m}}(y^{ ext{sat}}_{ ext{g}}) = rac{\lambda_{ ext{g}}}{D_e} rac{\Phi}{D_e} \quad \left[< h^{ ext{sat}}_{ ext{g}} - h^{ ext{sat}}_{ ext{l}}
ight]$$
$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} < h_{\vartheta}^{\scriptscriptstyle \mathrm{aat}} - h_{\varrho}^{\scriptscriptstyle \mathrm{aat}}$$

Summary of the observations:

- Mixture presence
- Position y_{ℓ}^{salt} implicitly defined by $h_{\ell}(y_{\ell}^{\text{salt}}) = h_{\ell}^{\text{salt}}$ and we have $(h_{\ell})'(y_{\ell}^{\text{salt}}) = 0$
- Position $y_{\vartheta}^{\scriptscriptstyle Aat}$ computed w.r.t. $y_{\ell}^{\scriptscriptstyle Aat}$ by $y_{\vartheta}^{\scriptscriptstyle Aat} = y_{\ell}^{\scriptscriptstyle Aat} + \frac{D_e}{\Phi}(h_{\vartheta}^{\scriptscriptstyle Aat} h_{\ell}^{\scriptscriptstyle Aat}) \frac{\lambda_{\vartheta}}{D_e}$

• Gas diffusion reduces mixture region for steady solution $(y_g^{\text{saft}} - y_l^{\text{saft}}) = (x_g^{\text{saft}} - x_l^{\text{saft}}) - rac{\lambda_g}{D_e}$

Jump occurs within the mixture region and

$$\llbracket h
rbracket(y^{ ext{salt}}_{ ext{g}}) = h^{ ext{salt}}_{ ext{g}} - h_{ ext{m}}(y^{ ext{salt}}_{ ext{g}}) = rac{\lambda_{ ext{g}}}{D_e} rac{\Phi}{D_e} \quad \left[< h^{ ext{salt}}_{ ext{g}} - h^{ ext{salt}}_{ ext{g}}
ight]$$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} < h_{\vartheta}^{\scriptscriptstyle \mathrm{aat}} - h_{\varrho}^{\scriptscriptstyle \mathrm{aat}}$$

Summary of the observations:

- Mixture presence
- Position y_{ℓ}^{sat} implicitly defined by $h_{\ell}(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}}$ and we have $(h_{\ell})'(y_{\ell}^{\text{sat}}) = 0$
- Position $y_{\vartheta}^{\scriptscriptstyle \text{soft}}$ computed w.r.t. $y_{\varrho}^{\scriptscriptstyle \text{soft}}$ by $y_{\vartheta}^{\scriptscriptstyle \text{soft}} = y_{\varrho}^{\scriptscriptstyle \text{soft}} + \frac{D_e}{\Phi}(h_{\vartheta}^{\scriptscriptstyle \text{soft}} h_{\varrho}^{\scriptscriptstyle \text{soft}}) \frac{\lambda_{\vartheta}}{D_e}$
- Gas diffusion reduces mixture region for steady solution $(y_{g}^{\text{saf}} y_{\ell}^{\text{saf}}) = (x_{g}^{\text{saf}} x_{\ell}^{\text{saf}}) \frac{\lambda_{g}}{D_{e}}$
- Jump occurs within the mixture region and

$$\llbracket h
rbracket(y^{ ext{sat}}_{ ext{g}}) = h^{ ext{sat}}_{ ext{g}} - h_{ ext{m}}(y^{ ext{sat}}_{ ext{g}}) = rac{\lambda_{ ext{g}}}{D_e} rac{\Phi}{D_e} \quad \left[< h^{ ext{sat}}_{ ext{g}} - h^{ ext{sat}}_{ ext{l}}
ight]$$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} < h_{\vartheta}^{\scriptscriptstyle \mathrm{aat}} - h_{\varrho}^{\scriptscriptstyle \mathrm{aat}}$$

Summary of the observations:

- Mixture presence
- Position $y_{\ell}^{\scriptscriptstyle \rm sat}$ implicitly defined by $h_{\ell}(y_{\ell}^{\scriptscriptstyle \rm sat}) = h_{\ell}^{\scriptscriptstyle \rm sat}$ and we have $(h_{\ell})'(y_{\ell}^{\scriptscriptstyle \rm sat}) = 0$
- Position $y_{\vartheta}^{\scriptscriptstyle \text{soft}}$ computed w.r.t. $y_{\varrho}^{\scriptscriptstyle \text{soft}}$ by $y_{\vartheta}^{\scriptscriptstyle \text{soft}} = y_{\varrho}^{\scriptscriptstyle \text{soft}} + \frac{D_e}{\Phi}(h_{\vartheta}^{\scriptscriptstyle \text{soft}} h_{\varrho}^{\scriptscriptstyle \text{soft}}) \frac{\lambda_{\vartheta}}{D_e}$
- Gas diffusion reduces mixture region for steady solution $(y_{g}^{\text{sat}} y_{\ell}^{\text{sat}}) = (x_{g}^{\text{sat}} x_{\ell}^{\text{sat}}) \frac{\lambda_{g}}{D_{e}}$
- Jump occurs within the mixture region and

$$\llbracket h
rbracket(y^{ ext{sat}}_{2}) = h^{ ext{sat}}_{2} - h_{ ext{m}}(y^{ ext{sat}}_{2}) = rac{\lambda_{ ext{g}}}{D_{e}} rac{\Phi}{D_{e}} \quad \left[< h^{ ext{sat}}_{2} - h^{ ext{sat}}_{ ext{l}}
ight]$$

$$\mathsf{Case}\;rac{\lambda_{artheta}}{D_{e}}rac{\Phi}{D_{e}} < h_{artheta}^{ ext{sat}} - h_{arlet}^{ ext{sat}}$$

Proposition 3

The mixture zone is present and the unique solution is discontinuous at y_q^{sal} and can be written as

$$h(y) = \begin{cases} h_{\ell}(y) \stackrel{\text{def}}{=} C_{\ell,1} + \frac{\Phi}{D_e} y + C_{\ell,2} \exp\left(\frac{y}{\lambda_{\ell}/D_e}\right) & \text{if } y \leq y_{\ell}^{\text{sol}} \\ \\ h_{\mathcal{m}}(y) \stackrel{\text{def}}{=} h_{\ell}^{\text{sol}} + \frac{\Phi}{D_e} (y - y_{\ell}^{\text{sol}}) & \text{if } y_{\ell}^{\text{sol}} \leq y < y_{g}^{\text{sol}} \\ \\ h_{g}(y) \stackrel{\text{def}}{=} h_{g}^{\text{sol}} + \frac{\Phi}{D_e} (y - y_{g}^{\text{sol}}) & \text{if } y \geq y_{g}^{\text{sol}} \end{cases}$$

• The constants
$$C_{\ell,1}$$
 and $C_{\ell,2}$ depend on y_{ℓ}^{sat} .
The position y_{ℓ}^{sat} is implicitly defined by $h_{\ell}(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}}$ and $h'_{\ell}(y_{\ell}^{\text{sat}}) = 0$.

• The position $y_{g}^{\scriptscriptstyle aat}$ is computed w.r.t. $y_{\ell}^{\scriptscriptstyle aat}$ by $y_{g}^{\scriptscriptstyle aat} = y_{\ell}^{\scriptscriptstyle aat} + \frac{D_{e}}{\Phi}(h_{g}^{\scriptscriptstyle aat} - h_{\ell}^{\scriptscriptstyle aat}) - \frac{\lambda_{g}}{D_{e}}$.

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In pure phase regions, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa}/D_e}\right)$

- The boundary conditions give two relations:
 - Liquid region: $h_{l}(0)=h_{e}\rightsquigarrow G_{l,2}=h_{e}-G_{l,1}$
 - Vapor region: $\lim_{y o\infty} h_a'(y) = rac{w}{D_{\pi^*}} \rightsquigarrow C_{g,2} = 0$.
- In mixture region, we solve $D_e h'_m(y) = \Phi$, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e}y$
- We need to compute $C_{l,1}$, C_m , $C_{g,1}$ and the transition points y_l^{soft} , y_g^{soft} .

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In pure phase regions, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa/D_e}}\right)$

- The boundary conditions give two relations:
 - Liquid region: $n_{\rm I}(0) \equiv n_a \sim O_{\rm I,2} \equiv n_a O_{\rm I,1}$
- In mixture region, we solve $D_e h'_m(y) = \Phi$, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e}y$
- We need to compute $C_{\ell,1}$, C_m , $C_{g,1}$ and the transition points y_{l}^{sat} , y_{g}^{sat} .

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

- In pure phase regions, we solve $D_e h'_{\kappa}(y) \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa/D_e}}\right)$
- The boundary conditions give two relations:
 - Liquid region: $h_{\ell}(0)=h_e \rightsquigarrow C_{\ell,2}=h_e-C_{\ell,2}$
 - Vapor region: $\lim_{y o\infty}h'_g(y)=rac{\Phi}{D_e} imes C_{g,2}=0$
- In mixture region, we solve $D_e h'_m(y) = \Phi$, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e} y$

• We need to compute $C_{\ell,1}$, C_m , $C_{g,1}$ and the transition points y_{ℓ}^{saft} , y_{q}^{saft} .

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In pure phase regions, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa/D_e}}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0) = h_e \rightsquigarrow C_{\ell,2} = h_e - C_{\ell,2}$$

• Vapor region: $\lim_{y\to\infty} h'_g(y) = \frac{\Phi}{D_e} \to C_{g,2} = 0$

• In mixture region, we solve
$$D_e h'_m(y) = \Phi$$
, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e} y$

• We need to compute $C_{l,1}$, C_m , $C_{g,1}$ and the transition points y_l^{saft} , y_g^{saft} .

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In pure phase regions, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa/D_e}}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0) = h_e \rightsquigarrow C_{\ell,2} = h_e - C_{\ell,2}$$

• Vapor region: $\lim_{y\to\infty} h'_{\mathfrak{g}}(y) = \frac{\Phi}{D_e} \rightsquigarrow C_{\mathfrak{g},2} = 0$

• In mixture region, we solve $D_e h'_m(y) = \Phi$, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e}y$

ullet We need to compute $C_{[\![,1]}$, C_{m} , $C_{g,1}$ and the transition points $y_{[\![}^{\scriptscriptstyle halt}$, $y_{g}^{\scriptscriptstyle halt}$.

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In pure phase regions, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa/D_e}}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0) = h_e \rightsquigarrow C_{\ell,2} = h_e - C_{\ell,2}$$

• Vapor region: $\lim_{y\to\infty} h'_{\mathfrak{g}}(y) = \frac{\Phi}{D_e} \rightsquigarrow C_{\mathfrak{g},2} = 0$

• In mixture region, we solve $D_e h'_m(y) = \Phi$, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e} y$

ullet We need to compute $C_{eta,1}$, C_{m} , $C_{g,1}$ and the transition points $y_{eta}^{ ext{Adt}}$, $y_{a}^{ ext{Adt}}$

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into three regions, ordered from low to high y values:

• In pure phase regions, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa/D_e}}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0) = h_e \rightsquigarrow C_{\ell,2} = h_e - C_{\ell,2}$$

- Vapor region: $\lim_{y\to\infty} h'_{\mathfrak{z}}(y) = \frac{\Phi}{D_e} \rightsquigarrow C_{\mathfrak{z},2} = 0$
- In mixture region, we solve $D_e h'_m(y) = \Phi$, yielding $h_\kappa(y) = C_m + \frac{\Phi}{D_e}y$
- We need to compute $C_{\ell,1}$, C_m , $C_{g,1}$ and the transition points y_{ℓ}^{sat} , y_{g}^{sat} .

- Proof Liquid/Mixture transition.
 - Jump relation at $y_{\mathbb{l}}^{\scriptscriptstyle{\mathrm{adt}}}$:

$$D_{e}\llbracket h(y_{\ell}^{\text{sat}}) \rrbracket - 0 \times h'(y_{\ell}^{\text{sat},+}) + \lambda_{\ell} h'(y_{\ell}^{\text{sat},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\ell} h'(y_{\ell}^{\text{sat},-})}_{\geq 0} = \underbrace{-D_{e}}_{<0} \underbrace{\llbracket h(y_{\ell}^{\text{sat}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \begin{cases} \llbracket h(y_{\ell}^{\text{sat}}) \rrbracket = 0 \\ h'(y_{\ell}^{\text{sat},-}) = 0 \end{cases}$$

- Continuity:
 - $h_{\ell}(y_{\ell}^{ad}) = h_{\ell}^{ad} \sim G_{l,1}$ is a function of y_{ℓ}^{ad} • $h_{m}(y_{\ell}^{ad}) = h_{\ell}^{ad} \sim G_{m}$ is a function of y_{ℓ}^{ad}
- Slope in the liquid region:
 - . The position $y_{
 m f}^{
 m add}$ is implicitly defined by $h'(y_{
 m f}^{
 m add})=0$.

- Proof Liquid/Mixture transition.
 - Jump relation at y_{l}^{sat} :

$$D_{e}\llbracket h(y_{\mathbb{I}}^{\text{salt}}) \rrbracket - 0 \times h'(y_{\mathbb{I}}^{\text{salt},+}) + \lambda_{\mathbb{I}} h'(y_{\mathbb{I}}^{\text{salt},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\mathbb{I}} h'(y_{\mathbb{I}}^{\text{salt},-})}_{\geq 0} = \underbrace{-D_{e}}_{<0} \underbrace{\llbracket h(y_{\mathbb{I}}^{\text{salt}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \begin{cases} \llbracket h(y_{\mathbb{I}}^{\text{salt}}) \rrbracket = 0 \\ h'(y_{\mathbb{I}}^{\text{salt},-}) = 0 \end{cases}$$

- Continuity:
 - $h_{\ell}(y_{\ell}^{\text{salt}}) = h_{\ell}^{\text{salt}} \rightsquigarrow C_{\ell,1}$ is a function of y_{ℓ}^{salt}
 - $n_{\mathrm{m}}(g_{\parallel}) = n_{\parallel} \rightarrow C_{\mathrm{m}}$ is a function
- Slope in the liquid region:
 - The position $y_{
 m f}^{
 m and}$ is implicitly defined by $h'(y_{
 m f}^{
 m and})=0$.

- Proof Liquid/Mixture transition.
 - Jump relation at y_{l}^{sat} :

$$D_{e}\llbracket h(y_{\mathbb{I}}^{\text{sal}}) \rrbracket - 0 \times h'(y_{\mathbb{I}}^{\text{sal},+}) + \lambda_{\mathbb{I}} h'(y_{\mathbb{I}}^{\text{sal},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\mathbb{I}} h'(y_{\mathbb{I}}^{\text{sal},-})}_{\geq 0} = \underbrace{-D_{e}}_{<0} \underbrace{\llbracket h(y_{\mathbb{I}}^{\text{sal}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \begin{cases} \llbracket h(y_{\mathbb{I}}^{\text{sal}}) \rrbracket = 0 \\ h'(y_{\mathbb{I}}^{\text{sal},-}) = 0 \end{cases}$$

- Continuity:
 - $h_{\ell}(y_{\ell}^{\text{sal}}) = h_{\ell}^{\text{sal}} \rightsquigarrow C_{\ell,1}$ is a function of y_{ℓ}^{sal} • $h_m(y_{\ell}^{\text{sal}}) = h_{\ell}^{\text{sal}} \rightsquigarrow C_m$ is a function of y_{ℓ}^{sal}
- Slope in the liquid region:
 - . The position $y_{
 m f}^{
 m ad}$ is implicitly defined by $h'(y_{
 m f}^{
 m add})=0$.

- Proof Liquid/Mixture transition.
 - Jump relation at y_{l}^{sat} :

$$D_{e}\llbracket h(y_{\ell}^{\text{salt}}) \rrbracket - 0 \times h'(y_{\ell}^{\text{salt},+}) + \lambda_{\ell} h'(y_{\ell}^{\text{salt},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\ell} h'(y_{\ell}^{\text{salt},-})}_{\geq 0} = \underbrace{-D_{e}}_{<0} \underbrace{\llbracket h(y_{\ell}^{\text{salt}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \begin{cases} \llbracket h(y_{\ell}^{\text{salt}}) \rrbracket = 0 \\ h'(y_{\ell}^{\text{salt},-}) = 0 \end{cases}$$

- Continuity:
 - $h_{\ell}(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_{\ell,1}$ is a function of y_{ℓ}^{sat} • $h_m(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_m$ is a function of y_{ℓ}^{sat}
- Slope in the liquid region:
 - The position $y_1^{_{
 m boll}}$ is implicitly defined by $h'(y_1^{_{
 m boll}},{}^{-})=0$.

- Proof Liquid/Mixture transition.
 - Jump relation at y_{l}^{sat} :

$$D_{e}\llbracket h(y_{\ell}^{\text{salt}}) \rrbracket - 0 \times h'(y_{\ell}^{\text{salt},+}) + \lambda_{\ell} h'(y_{\ell}^{\text{salt},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\ell} h'(y_{\ell}^{\text{salt},-})}_{\geq 0} = \underbrace{-D_{e}}_{<0} \underbrace{\llbracket h(y_{\ell}^{\text{salt}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \begin{cases} \llbracket h(y_{\ell}^{\text{salt}}) \rrbracket = 0 \\ h'(y_{\ell}^{\text{salt},-}) = 0 \end{cases}$$

- Continuity:
 - $h_{\ell}(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_{\ell,1}$ is a function of y_{ℓ}^{sat} • $h_m(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_m$ is a function of y_{ℓ}^{sat}
- Slope in the liquid region:
 - ullet The position $y_{eta}^{ imes lpha t}$ is implicitly defined by $h'(y_{eta}^{ imes lpha t},-)=0$

- Proof Liquid/Mixture transition.
 - Jump relation at y_{l}^{sat} :

$$D_{e}\llbracket h(y_{\ell}^{\text{salt}}) \rrbracket - 0 \times h'(y_{\ell}^{\text{salt},+}) + \lambda_{\ell} h'(y_{\ell}^{\text{salt},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\ell} h'(y_{\ell}^{\text{salt},-})}_{\geq 0} = \underbrace{-D_{e}}_{<0} \underbrace{\llbracket h(y_{\ell}^{\text{salt}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \begin{cases} \llbracket h(y_{\ell}^{\text{salt}}) \rrbracket = 0 \\ h'(y_{\ell}^{\text{salt},-}) = 0 \end{cases}$$

- Continuity:
 - $h_{\ell}(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_{\ell,1}$ is a function of y_{ℓ}^{sat} • $h_m(y_{\ell}^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_m$ is a function of y_{ℓ}^{sat}
- Slope in the liquid region:
 - The position $y_{\emptyset}^{\mathrm{sat}}$ is implicitly defined by $h'(y_{\emptyset}^{\mathrm{sat},-})=0$

- ${\sf Proof-Mixture}/{\sf Gas\ transition}.$
 - Jump relation at y_{g}^{saft} :

$$D_e \llbracket h(y_{\mathfrak{f}}^{\text{sat}}) \rrbracket - \lambda_{\mathfrak{f}} h'(y_{\mathfrak{f}}^{\text{sat},+}) + 0 \times h'(y_{\mathfrak{f}}^{\text{sat},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\mathfrak{f}} h'(y_{\mathfrak{f}}^{\text{sat},+})}_{\geq 0} = \underbrace{D_e}_{>0} \underbrace{\llbracket h(y_{\mathfrak{f}}^{\text{sat}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \llbracket h(y_{\mathfrak{f}}^{\text{sat}}) \rrbracket = \frac{\lambda_{\mathfrak{f}}}{D_e} \frac{\Phi}{D_e}$$

• An infinite number of solutions

$$h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\mathtt{aal}}) \in \left[h_{\mathfrak{g}}^{\mathtt{aal}}, h_{\mathfrak{g}}^{\mathtt{aal}} + \frac{\lambda_{\mathfrak{g}}}{D_{e}} \frac{\Phi}{D_{e}} \left[\quad \rightsquigarrow \quad h_{\mathfrak{m}}(y_{\mathfrak{g}}^{\mathtt{aal}}) = h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\mathtt{aal}}) - \llbracket h(y_{\mathfrak{g}}^{\mathtt{aal}}) \rrbracket \right]$$

• The viscosity solution corresponds to the smallest mixture region: $h_g(y_g^{\text{sat}}) = h_g^{\text{sat}}$ • This fixes y_g^{sat} and thus $C_{g,1}$.

- Proof Mixture/Gas transition.
 - Jump relation at y_{g}^{saft} :

$$D_e \llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket - \lambda_{\mathfrak{g}} h'(y_{\mathfrak{g}}^{\text{soft},+}) + 0 \times h'(y_{\mathfrak{g}}^{\text{soft},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\mathfrak{g}} h'(y_{\mathfrak{g}}^{\text{soft},+})}_{\geq 0} = \underbrace{D_e}_{>0} \underbrace{\llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket = \frac{\lambda_{\mathfrak{g}}}{D_e} \underbrace{\Phi}_{D_e}$$

• An infinite number of solutions

$$h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\mathtt{aat}}) \in \left[h_{\mathfrak{g}}^{\mathtt{aat}}, h_{\mathfrak{g}}^{\mathtt{aat}} + \frac{\lambda_{\mathfrak{g}}}{D_{e}} \frac{\Phi}{D_{e}} \left[\quad \rightsquigarrow \quad h_{\mathfrak{m}}(y_{\mathfrak{g}}^{\mathtt{aat}}) = h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\mathtt{aat}}) - \llbracket h(y_{\mathfrak{g}}^{\mathtt{aat}}) \rrbracket \right]$$

• The viscosity solution corresponds to the smallest mixture region: $h_g(y_g^{\text{solt}}) = h_g^{\text{solt}}$ • This fixes y_g^{solt} and thus $C_{g,1}$.

- Proof Mixture/Gas transition.
 - Jump relation at y_{g}^{saft} :

$$D_e \llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket - \lambda_{\mathfrak{g}} h'(y_{\mathfrak{g}}^{\text{soft},+}) + 0 \times h'(y_{\mathfrak{g}}^{\text{soft},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\mathfrak{g}} h'(y_{\mathfrak{g}}^{\text{soft},+})}_{\geq 0} = \underbrace{D_e}_{>0} \underbrace{\llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket = \frac{\lambda_{\mathfrak{g}}}{D_e} \underbrace{\Phi}_{D_e}$$

• An infinite number of solutions

$$h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\scriptscriptstyle \mathrm{adt}}) \in \left[h_{\mathfrak{g}}^{\scriptscriptstyle \mathrm{adt}}, h_{\mathfrak{g}}^{\scriptscriptstyle \mathrm{adt}} + rac{\lambda_{\mathfrak{g}}}{D_e} rac{\Phi}{D_e} \left[\quad \leadsto \quad h_{\scriptscriptstyle m}(y_{\mathfrak{g}}^{\scriptscriptstyle \mathrm{adt}}) = h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\scriptscriptstyle \mathrm{adt}}) - \llbracket h(y_{\mathfrak{g}}^{\scriptscriptstyle \mathrm{adt}})
ight]$$

• The viscosity solution corresponds to the smallest mixture region: $h_g(y_g^{\text{sat}}) = h_g^{\text{sat}}$ • This fixes y_q^{sal} and thus $C_{g,1}$.

- Proof Mixture/Gas transition.
 - Jump relation at $y_{\mathfrak{R}}^{\mathrm{sat}}$:

$$D_e \llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket - \lambda_{\mathfrak{g}} h'(y_{\mathfrak{g}}^{\text{soft},+}) + 0 \times h'(y_{\mathfrak{g}}^{\text{soft},-}) = 0 \quad \rightsquigarrow \quad \underbrace{\lambda_{\mathfrak{g}} h'(y_{\mathfrak{g}}^{\text{soft},+})}_{\geq 0} = \underbrace{D_e}_{>0} \underbrace{\llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket}_{\geq 0} \quad \rightsquigarrow \quad \llbracket h(y_{\mathfrak{g}}^{\text{soft}}) \rrbracket = \frac{\lambda_{\mathfrak{g}}}{D_e} \underbrace{\Phi}_{D_e}$$

• An infinite number of solutions

$$h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\mathtt{aat}}) \in \left[h_{\mathfrak{g}}^{\mathtt{aat}}, h_{\mathfrak{g}}^{\mathtt{aat}} + \frac{\lambda_{\mathfrak{g}}}{D_{e}} \frac{\Phi}{D_{e}} \left[\quad \rightsquigarrow \quad h_{\mathfrak{m}}(y_{\mathfrak{g}}^{\mathtt{aat}}) = h_{\mathfrak{g}}(y_{\mathfrak{g}}^{\mathtt{aat}}) - \llbracket h(y_{\mathfrak{g}}^{\mathtt{aat}}) \rrbracket \right]$$

• The viscosity solution corresponds to the smallest mixture region: $h_g(y_g^{\rm salt}) = h_g^{\rm salt}$

 \bullet This fixes $y_{\theta}^{\rm salt}$ and thus $C_{\theta^{,1}}.$

Solution in the sense of distribution

No need viscosity solution when $\lambda_m = 0$ and uniqueness of the continuous solution when $\lambda_m > 0$

 $D_e h' - (L \circ h)'' = \Phi$ with $h \mapsto L(h)$ which has the graph

Suppose that $\llbracket h \rrbracket(y_*) > 0$ at a point y_* . What about $(L \circ h)(y_*)$ and y_* ?

We prove that
$$\llbracket L \circ h \rrbracket(y_*) = 0$$
:

$$\begin{cases} \llbracket h \rrbracket(y_*) > 0 \\ \llbracket L \circ h \rrbracket(y_*) > 0 \end{cases} \implies \begin{cases} h' \text{ contains a } \delta_{y_*} \\ (L \circ h)' \text{ contains a } \delta_{y_*} \end{cases}$$

 $\begin{cases} h' \text{ order } 0 \\ (L \circ h)'' \text{ order } 1 \end{cases}$

Since $L(h(y_*^+)) = L(h(y_*^-))$, this implies $h_{\ell}^{\text{adt}} \leq h(y_*^-) < h(y_*^+) \leq h_{\theta}^{\text{adt}}$ Note that, if $\lambda_m > 0$, we have $[\![L \circ h]\!](y_*) = 0$ iff $[\![h]\!](y_*) = 0$: jumps are not allowed

• What about $\llbracket (L \circ h)' \rrbracket (y_*)$?

$$\begin{cases} \llbracket h \rrbracket(y_*) > 0 \\ D_e h - (L \circ h)' \text{ continuous} \end{cases} \implies \llbracket (L \circ h)' \rrbracket(y_*) > 0 \implies h(y_*^+) = h_{\vartheta}^{\text{sal}} \text{ or } h(y_*^-) = h_{\varrho}^{\text{sal}} \end{cases}$$

Solution in the sense of distribution

No need viscosity solution when $\lambda_m=0$ and uniqueness of the continuous solution when $\lambda_m>0$

$$D_e h' - (L \circ h)'' = \Phi$$
 with $h \mapsto L(h)$ which has the graph

Suppose that $\llbracket h \rrbracket(y_*) > 0$ at a point y_* . What about $(L \circ h)(y_*)$ and y_* ?

• We prove that $\llbracket L \circ h \rrbracket(y_*) = 0$:

$$\begin{cases} \llbracket h \rrbracket(y_*) > 0 \\ \llbracket L \circ h \rrbracket(y_*) > 0 \end{cases} \implies \begin{cases} h' \text{ contains a } \delta_{y_*} \\ (L \circ h)' \text{ contains a } \delta_{y_*} \end{cases}$$



Since $L(h(y^+_*)) = L(h(y^-_*))$, this implies $h_{\ell}^{\text{solt}} \leq h(y^-_*) < h(y^+_*) \leq h_{\vartheta}^{\text{solt}}$ Note that, if $\lambda_m > 0$, we have $[\![L \circ h]\!](y_*) = 0$ iff $[\![h]\!](y_*) = 0$: jumps are not allowed.

• What about $\llbracket (L \circ h)' \rrbracket (y_*)$?

 $\begin{cases} \llbracket h \rrbracket(y_*) > 0 \\ D_e h - (L \circ h)' \text{ continuous} \end{cases} \implies \llbracket (L \circ h)' \rrbracket(y_*) > 0 \implies h(y_*^+) = h_{g}^{\text{adt}} \text{ or } h(y_*^-) = h_{\ell}^{\text{adt}} \end{cases}$

Solution in the sense of distribution

No need viscosity solution when $\lambda_m = 0$ and uniqueness of the continuous solution when $\lambda_m > 0$

$$D_e h' - (L \circ h)'' = \Phi$$
 with $h \mapsto L(h)$ which has the graph

Suppose that $\llbracket h \rrbracket(y_*) > 0$ at a point y_* . What about $(L \circ h)(y_*)$ and y_* ?

• We prove that $\llbracket L \circ h \rrbracket(y_*) = 0$:

$$\begin{cases} \llbracket h \rrbracket(y_*) > 0 \\ \llbracket L \circ h \rrbracket(y_*) > 0 \end{cases} \implies \begin{cases} h' \text{ contains a } \delta_{y_*} \\ (L \circ h)' \text{ contains a } \delta_{y_*} \end{cases} \implies \begin{cases} h' \text{ order } 0 \\ (L \circ h)'' \text{ order } 1 \end{cases} \notin$$



• What about $\llbracket (L \circ h)' \rrbracket (y_*)$?

$$\begin{cases} \llbracket h \rrbracket(y_*) > 0 \\ D_e h - (L \circ h)' \text{ continuous} \end{cases} \implies \llbracket (L \circ h)' \rrbracket(y_*) > 0 \implies h(y_*^+) = h_{\vartheta}^{\text{\tiny add}} \text{ or } h(y_*^-) = h_{\vartheta}^{\text{\tiny add}} \end{cases}$$

 h_{a}^{sal}

Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_{g} > 0$ (degenerate diffusion)

What about if the condition
$$rac{\lambda_{artheta}}{D_e}rac{\Phi}{D_e} \leq h_{artheta}^{\scriptscriptstyle{
m solt}} - h_{arlepsilon}^{\scriptscriptstyle{
m solt}}$$
 is not met?

A direct transition from liquid to gas must be considered. It is a sharp interface model since no mixture region is present. Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_{g} > 0$ (degenerate diffusion)

What about if the condition
$$rac{\lambda_{artheta}}{D_e}rac{\Phi}{D_e}\leq h_{artheta}^{\scriptscriptstyle{
m act}}-h_{arleple}^{\scriptscriptstyle{
m act}}$$
 is not met?

A direct transition from liquid to gas must be considered. It is a sharp interface model since no mixture region is present.

$$\mathsf{Case} \ \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} \geq h_{\vartheta}^{\mathrm{act}} - h_{\varrho}^{\mathrm{act}}$$

Summary of observations:

- Mixture does not exist
- If $\lambda_{\ell} > 0$, the jump is constant $\llbracket h
 rbracket(y^{\text{sat}}) = h_{\mathfrak{g}}(y^{\text{sat}}) h_{\ell}(y^{\text{sat}}) = h_{\mathfrak{g}}^{\text{sat}} h_{\ell}^{\text{sat}}$
- ullet Position $y^{
 m sat}=y^{
 m sat}_{ar l}=y^{
 m sat}_{g}$ is implicitly defined by $h_{ar l}(y^{
 m sat})=h^{
 m sat}_{ar l}$ with

$$(h_{\ell})'(y^{\text{sal}}) = \frac{\frac{\lambda_{g}}{D_{e}} \frac{\Phi}{D_{e}} - (h_{g}^{\text{sal}} - h_{\ell}^{\text{sal}})}{\frac{\lambda_{\ell}}{D_{e}}} \qquad \left[\geq 0 \text{ and } \frac{1}{\lambda_{\ell} \to 0} + \infty \right]$$

• If $\lambda_\ell = 0$, the jump occurs within the liquid region and $h_{\ell}(y^{\text{sat}}) < h_{\ell}^{\text{sat}}$:

$$\llbracket h \rrbracket(y^{\text{sal}}) = h_{\theta}^{\text{sal}} - h_{\ell}(y^{\text{sal}}) = \frac{\lambda_{\theta}}{D_e} \frac{\Phi}{D_e} \ge h_{\theta}^{\text{sal}} - h_{\ell}^{\text{sal}}$$

$$\mathsf{Case} \ \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} \geq h_{\vartheta}^{\mathrm{sat}} - h_{\ell}^{\mathrm{sat}}$$

Summary of observations:

- Mixture does not exist
- If $\lambda_{\ell} > 0$, the jump is constant $\llbracket h
 rbracket(y^{\text{sat}}) = h_{g}(y^{\text{sat}}) h_{\ell}(y^{\text{sat}}) = h_{g}^{\text{sat}} h_{\ell}^{\text{sat}}$

• Position $y^{_{sal}}=y^{_{aal}}_{l}=y^{_{sal}}_{q}$ is implicitly defined by $h_{l}(y^{_{sal}})=h^{_{sal}}_{l}$ with

$$(h_{\ell})'(y^{\text{sak}}) = \frac{\frac{\lambda_{\theta}}{D_e} \frac{\Phi}{D_e} - (h_{\theta}^{\text{sak}} - h_{\ell}^{\text{sak}})}{\frac{\lambda_{\ell}}{D_e}} \qquad \left[\ge 0 \text{ and } \frac{1}{\lambda_{\ell} \to 0} + \infty \right]$$

• If $\lambda_\ell=0$, the jump occurs within the liquid region and $h_{\ell}(y^{\text{sat}}) < h_{\ell}^{\text{sat}}$:

$$\llbracket h \rrbracket(y^{\text{sal}}) = h_{\theta}^{\text{sal}} - h_{l}(y^{\text{sal}}) = \frac{\lambda_{\theta}}{D_{e}} \frac{\Phi}{D_{e}} \geq h_{\theta}^{\text{sal}} - h_{l}^{\text{sal}}$$

$$\mathsf{Case} \ \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} \geq h_{\vartheta}^{\mathrm{sat}} - h_{\ell}^{\mathrm{sat}}$$

Summary of observations:

- Mixture does not exist
- If $\lambda_{\ell} > 0$, the jump is constant $\llbracket h \rrbracket(y^{\text{sat}}) = h_{g}(y^{\text{sat}}) h_{\ell}(y^{\text{sat}}) = h_{g}^{\text{sat}} h_{\ell}^{\text{sat}}$
- Position $y^{\scriptscriptstyle \rm sal} = y^{\scriptscriptstyle \rm sal}_{\ell} = y^{\scriptscriptstyle \rm sal}_{\beta}$ is implicitly defined by $h_{\ell}(y^{\scriptscriptstyle \rm sal}) = h^{\scriptscriptstyle \rm sal}_{\ell}$ with

$$\left(h_{\ell}
ight)'\left(y^{\scriptscriptstyle{ ext{sat}}}
ight) = rac{rac{\lambda_{artheta}}{D_{e}}rac{\Phi}{D_{e}}-\left(h_{artheta}^{\scriptscriptstyle{ ext{sat}}}-h_{\ell}^{\scriptscriptstyle{ ext{sat}}}
ight)}{rac{\lambda_{\ell}}{D_{e}}} \qquad \left[\geq 0 ext{ and } rac{\lambda_{\ell} o 0}{\lambda_{\ell} o 0} +\infty
ight]$$

• If $\lambda_\ell=0$, the jump occurs within the liquid region and $h_{\ell}(y^{\text{sat}}) < h_{\ell}^{\text{sat}}$:

$$\llbracket h \rrbracket(y^{\text{mat}}) = h^{\text{mat}}_{\theta} - h_{\mathbb{I}}(y^{\text{mat}}) = \frac{\lambda_{\theta}}{D_e} \frac{\Phi}{D_e} \geq h^{\text{mat}}_{\theta} - h^{\text{mat}}_{\mathbb{I}}$$

$$\mathsf{Case} \ \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} \geq h_{\vartheta}^{\mathrm{sat}} - h_{\ell}^{\mathrm{sat}}$$

Summary of observations:

- Mixture does not exist
- If $\lambda_{\ell} > 0$, the jump is constant $\llbracket h \rrbracket(y^{\text{sat}}) = h_{\theta}(y^{\text{sat}}) h_{\ell}(y^{\text{sat}}) = h_{\theta}^{\text{sat}} h_{\ell}^{\text{sat}}$
- Position $y^{\scriptscriptstyle \rm sal} = y^{\scriptscriptstyle \rm sal}_{\ell} = y^{\scriptscriptstyle \rm sal}_{\beta}$ is implicitly defined by $h_{\ell}(y^{\scriptscriptstyle \rm sal}) = h^{\scriptscriptstyle \rm sal}_{\ell}$ with

$$(h_{\ell})'(y^{\scriptscriptstyle \mathrm{sat}}) = rac{rac{\lambda_{ heta}}{D_e} rac{\Phi}{D_e} - (h_{ heta}^{\scriptscriptstyle \mathrm{sat}} - h_{\ell}^{\scriptscriptstyle \mathrm{sat}})}{rac{\lambda_{\ell}}{D_e}} \qquad \left[\ge 0 \; \mathsf{and} \; rac{\lambda_{\ell}
ightarrow 0}{\lambda_{\ell}
ightarrow 0} + \infty
ight]$$

• If $\lambda_\ell = 0$, the jump occurs within the liquid region and $h_{\ell}(y^{\text{sat}}) < h_{\ell}^{\text{sat}}$:

$$\llbracket h
rbracket(y^{ ext{sat}}) = h^{ ext{sat}}_{ ext{g}} - h_{\ell}(y^{ ext{sat}}) = rac{\lambda_{ ext{g}}}{D_e} rac{\Phi}{D_e} \geq h^{ ext{sat}}_{ ext{g}} - h^{ ext{sat}}_{\ell}$$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} \geq h_{\vartheta}^{\scriptscriptstyle{\mathrm{aat}}} - h_{\varrho}^{\scriptscriptstyle{\mathrm{aat}}}$$

Proposition 4

The mixture zone is not present and the unique solution is discontinuous and can be written as

$$h(y) = \begin{cases} h_{\ell}(y) \stackrel{\text{def}}{=} C_{\ell,1} + \frac{\Phi}{D_e} y + C_{\ell,2} \exp\left(\frac{y}{\lambda_{\ell}/D_e}\right) & \text{if } y \le y^{\text{sal}} \\ \\ h_{g}(y) \stackrel{\text{def}}{=} h_{g}^{\text{sal}} + \frac{\Phi}{D_e} (y - y^{\text{sal}}) & \text{if } y \ge y^{\text{sal}} \end{cases}$$

• The constants $C_{\ell,1}$ and $C_{\ell,2}$ depend on y^{sal} .

• The position y^{sal} is implicitly defined by $D_e \left(h_{\mathcal{J}}^{\text{sal}} - h_{\ell}^{\text{sal}} \right) = \lambda_{\mathcal{J}} h'(y^{\text{sal},+}) - \lambda_{\ell} h'(y^{\text{sal},-})$

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into two regions, ordered from low to high y values:

• In each region, we solve $D_e h'_\kappa(y) - \lambda_\kappa h''_\kappa(y) = \Phi$, yielding $h_\kappa(y) = C_{\kappa,1} + rac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(rac{y}{\lambda_{y/D_e}}
ight)$

- The boundary conditions give two relations:
 - Liquid region: $h_{l}(0)=h_{e} \rightsquigarrow C_{l,2}=h_{e}-C_{l,1}$
 - Vapor region: $\lim_{y
 ightarrow\infty}h_q'(y)=rac{w}{D_c}\rightsquigarrow C_{g,2}=0$.
- We need to compute $C_{\ell,1}$ and $C_{g,1}$ and the transition point y^{saf} .

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into two regions, ordered from low to high y values:

$$\begin{matrix} & \mathsf{Liquid} & \mathsf{Gas} \\ 0 & \kappa = \mathbb{I} & y^{\mathrm{sat}} & \kappa = \frac{1}{\theta} \end{matrix}$$

• In each region, we solve
$$D_e h'_\kappa(y) - \lambda_\kappa h''_\kappa(y) = \Phi$$
, yielding $h_\kappa(y) = C_{\kappa,1} + rac{\Phi}{D_e}y + C_{\kappa,2}\exp\left(rac{y}{\lambda_{\|}/D_e}
ight)$

- The boundary conditions give two relations:
 - . Vapor region: $\lim_{y
 ightarrow\infty}h_g'(y)=rac{\Phi}{D_c}\sim C_{g,2}=0$.
- We need to compute $C_{\ell,1}$ and $C_{g,1}$ and the transition point y^{saf} .

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into two regions, ordered from low to high y values:

• In each region, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\mathbb{Q}}/D_e}\right)$

- The boundary conditions give two relations:
 - Liquid region: $h_{\ell}(0)=h_e \rightsquigarrow C_{\ell,2}=h_e-C_{\ell,2}$
 - Vapor region: $\lim_{y o\infty}h'_{g}(y)=rac{\Phi}{D_{e}} wo C_{g,2}=0$
- We need to compute $C_{\ell,1}$ and $C_{g,1}$ and the transition point $y^{\scriptscriptstyle hal}$

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into two regions, ordered from low to high y values:

$$\begin{matrix} & \mathsf{Liquid} & \mathsf{Gas} \\ 0 & \kappa = \mathbb{I} & y^{\mathrm{sat}} & \kappa = \frac{1}{g} \end{matrix}$$

• In each region, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\kappa}/D_e}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0)=h_e \rightsquigarrow C_{\ell,2}=h_e-C_{\ell,2}$$

• Vapor region: $\lim_{y\to\infty} h'_4(y) = \frac{h}{D_2} \rightarrow C_{4,2} = 0$

Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into two regions, ordered from low to high y values:

$$\begin{matrix} & \mathsf{Liquid} & \mathsf{Gas} \\ 0 & \kappa = \mathbb{I} & y^{\mathrm{sat}} & \kappa = \frac{1}{g} \end{matrix}$$

• In each region, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\|}/D_e}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0)=h_e \rightsquigarrow C_{\ell,2}=h_e-C_{\ell,1}$$

• Vapor region: $\lim_{y\to\infty}h'_{\mathfrak{g}}(y)=\frac{\Phi}{D_e}\rightsquigarrow C_{\mathfrak{g},2}=0$

ullet We need to compute $C_{\ell,1}$ and $C_{g,1}$ and the transition point $y^{\scriptscriptstyle bat}$
Proof – Solution on each region.

• Given $\Phi > 0$, $D_e > 0$, we observe that h increases. This leads to the division of space into two regions, ordered from low to high y values:

$$\begin{matrix} & \mathsf{Liquid} & \mathsf{Gas} \\ 0 & \kappa = \mathbb{I} & y^{\mathrm{sat}} & \kappa = \frac{1}{g} \end{matrix}$$

• In each region, we solve $D_e h'_{\kappa}(y) - \lambda_{\kappa} h''_{\kappa}(y) = \Phi$, yielding $h_{\kappa}(y) = C_{\kappa,1} + \frac{\Phi}{D_e} y + C_{\kappa,2} \exp\left(\frac{y}{\lambda_{\mathbb{Q}}/D_e}\right)$

• The boundary conditions give two relations:

• Liquid region:
$$h_{\ell}(0)=h_e \rightsquigarrow C_{\ell,2}=h_e-C_{\ell,1}$$

- Vapor region: $\lim_{y\to\infty} h'_{g}(y) = \frac{\Phi}{D_e} \rightsquigarrow C_{g,2} = 0$
- We need to compute $C_{\ell,1}$ and $C_{g,1}$ and the transition point y^{saft} .

Proof – Liquid/Gas transition.

This problem is a classic Stefan problem. Thus,

- The jump is fixed and equals the latent heat:
 - Liquid region: $h_{\emptyset}(y^{\text{sat}}) = h_{\emptyset}^{\text{sat}} \rightsquigarrow C_{\emptyset,1}$ is a function of y^{sat}
 - Vapor region: $h_g(y^{\text{solt}}) = h_a^{\text{solt}} \rightsquigarrow C_{g,1}$ is a function of y^{sol}

• The jump relation at y^{sat} is given by:

$$D_e \left(h_{\vartheta}^{\mathrm{sal}} - h_{\ell}^{\mathrm{sal}} \right) = \lambda_{\vartheta} h'(y^{\mathrm{sal},+}) - \lambda_{\ell} h'(y^{\mathrm{sal},-}) \quad \rightsquigarrow \quad \frac{\lambda_{\ell}}{D_e} h'_{\ell}(y^{\mathrm{sal},-}) = \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} - (h_{\vartheta}^{\mathrm{sal}} - h_{\ell}^{\mathrm{sal}}).$$

This relation provides an **implicit** definition of the position $y^{_{aab}}$ (which can be proven to be unique).

Proof – Liquid/Gas transition.

This problem is a classic Stefan problem. Thus,

- The jump is fixed and equals the latent heat:
 - Liquid region: $h_{\ell}(y^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_{\ell,1}$ is a function of y^{sat}
 - Vapor region: $h_q(y^{\text{ADL}}) = h_a^{\text{ADL}} \rightsquigarrow C_{q,1}$ is a function of y^{ADL}

• The jump relation at y^{Adt} is given by:

$$D_e \left(h_{\boldsymbol{\vartheta}}^{\mathrm{sat}} - h_{\boldsymbol{\ell}}^{\mathrm{sat}} \right) = \lambda_{\boldsymbol{\vartheta}} h'(\boldsymbol{y}^{\mathrm{sat},+}) - \lambda_{\boldsymbol{\ell}} h'(\boldsymbol{y}^{\mathrm{sat},-}) \quad \rightsquigarrow \quad \frac{\lambda_{\boldsymbol{\ell}}}{D_e} h'_{\boldsymbol{\ell}}(\boldsymbol{y}^{\mathrm{sat},-}) = \frac{\lambda_{\boldsymbol{\vartheta}}}{D_e} \frac{\Phi}{D_e} - (h_{\boldsymbol{\vartheta}}^{\mathrm{sat}} - h_{\boldsymbol{\ell}}^{\mathrm{sat}}).$$

This relation provides an **implicit** definition of the position $y^{_{aab}}$ (which can be proven to be unique).

Proof – Liquid/Gas transition.

This problem is a classic Stefan problem. Thus,

- The jump is fixed and equals the latent heat:
 - Liquid region: $h_{\ell}(y^{\text{sat}}) = h_{\ell}^{\text{sat}} \rightsquigarrow C_{\ell,1}$ is a function of y^{sat}
 - Vapor region: $h_g(y^{\text{solt}}) = h_g^{\text{solt}} \rightsquigarrow C_{g,1}$ is a function of y^{solt}

• The jump relation at y^{soft} is given by:

$$D_e \left(h_{\vartheta}^{\mathrm{salt}} - h_{\mathbb{l}}^{\mathrm{salt}} \right) = \lambda_{\vartheta} h'(y^{\mathrm{sal},+}) - \lambda_{\mathbb{l}} h'(y^{\mathrm{sal},-}) \quad \rightsquigarrow \quad \frac{\lambda_{\mathbb{l}}}{D_e} h'_{\mathbb{l}}(y^{\mathrm{sal},-}) = \frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} - (h_{\vartheta}^{\mathrm{salt}} - h_{\mathbb{l}}^{\mathrm{salt}}).$$

This relation provides an **implicit** definition of the position $y^{_{AOE}}$ (which can be proven to be unique).

Proof – Liquid/Gas transition.

This problem is a classic Stefan problem. Thus,

- The jump is fixed and equals the latent heat:
 - Liquid region: $h_{\ell}(y^{\text{salt}}) = h_{\ell}^{\text{salt}} \rightsquigarrow C_{\ell,1}$ is a function of y^{salt}
 - Vapor region: $h_g(y^{\text{sat}}) = h_g^{\text{sat}} \rightsquigarrow C_{g,1}$ is a function of y^{sat}
- The jump relation at y^{sat} is given by:

$$D_e \left(h_{ extsf{g}}^{ extsf{solt}} - h_{ extsf{l}}^{ extsf{solt}}
ight) = \lambda_{ extsf{g}} h'(y^{ extsf{solt},+}) - \lambda_{ extsf{l}} h'(y^{ extsf{solt},-}) \quad \rightsquigarrow \quad rac{\lambda_{ extsf{l}}}{D_e} h'_{ extsf{l}}(y^{ extsf{solt},-}) = rac{\lambda_{ extsf{g}}}{D_e} rac{\Phi}{D_e} - (h_{ extsf{g}}^{ extsf{solt},+} - h_{ extsf{l}}^{ extsf{solt},+}).$$

This relation provides an **implicit** definition of the position y^{sat} (which can be proven to be unique).

2. Steady-state model

- 2.1 The 1D steady-state model
- 2.2 Solution with $\lambda_{\ell} = \lambda_m = \lambda_q = 0$ (without diffusion)
- 2.3 Solution with $\lambda_0, \lambda_m, \lambda_q > 0$ (strictly positive diffusion)
- 2.4 Solution with $\lambda_m = 0$ and $\lambda_0, \lambda_a > 0$ (degenerate diffusion

2.5 Link to the Stationary Stefan Problem

2.6 Conclusion on the Steady-State Mode

Link to the Stationary Stefan Problem - free boundary problem

Find the position y^{aut} such that the over-determinate elliptic problem is satisfied:

Stationary Stefan problem on enthalpy

$$\begin{split} & \left[\begin{array}{l} D_e h' - \lambda_{\underline{l}} h'' = \Phi \\ h(0) = h_e < h_{\underline{l}}^{\mathrm{adl}}, & \quad \mathrm{in} \]0, y^{\mathrm{adl}} \\ h(y^{\mathrm{adl},-}) = h_{\underline{l}}^{\mathrm{adl}} \end{split} \right. \end{split}$$

$$egin{aligned} & D_e h' - \lambda_g h'' = \Phi \ & h(y^{ ext{ad},+}) = h_g^{ ext{ad}} & ext{in }]y^{ ext{ad}}, +\infty \ & \lim_{y o \infty} h'(y) = rac{\Phi}{D_e}, \end{aligned}$$

$$\lambda_{\boldsymbol{\vartheta}}\boldsymbol{h}'(\boldsymbol{y}^{\mathrm{sat}}) - \lambda_{\boldsymbol{\varrho}}\boldsymbol{h}'(\boldsymbol{y}^{\mathrm{sat}}) = D_{e}(\boldsymbol{h}_{\boldsymbol{\vartheta}}^{\mathrm{sat}} - \boldsymbol{h}_{\boldsymbol{\varrho}}^{\mathrm{sat}})$$

Remark: $\llbracket h(y^{\text{sat}}) \rrbracket = h_{\text{g}}^{\text{sat}} - h_{\mathbb{g}}^{\text{sat}}$

Stationary Stefan problem on temperature

$$\begin{cases} c_{p,l} D_e T' - \omega_l T'' = \Phi \\ T(0) = T_e < T^{\text{sal}}, \\ T(y^{\text{sal},-}) = T^{\text{sal}}, \end{cases} \quad \text{i}$$

in
$$]0,y^{\scriptscriptstyle{
m sat}}[$$

$$\begin{cases} c_{p,g} D_e T' - \omega_g T'' = \Phi \\ T(y^{\text{adt},+}) = T^{\text{adt}}, \\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g} D_e}, \end{cases}$$

 $\mid]y^{\scriptscriptstyle{ ext{sol}}},+\infty[$

$$\omega_{\mathrm{g}}T'(y^{\mathrm{sal},+})-\omega_{\mathrm{f}}T'(y^{\mathrm{sal},-})=D_{e}(h_{\mathrm{g}}^{\mathrm{sal}}-h_{\mathrm{f}}^{\mathrm{sal}})$$

Remark: $T(y^{\text{sal},-}) = T(y^{\text{sal},+}) = T^{\text{sal}}$

Link to the Stationary Stefan Problem - free boundary problem

Find the position $y^{\scriptscriptstyle{\rm Ault}}$ such that the over-determinate elliptic problem is satisfied:

Stationary Stefan problem on enthalpyStationary Stefan problem on temperature
$$\begin{cases} D_e h' - \lambda_{\ell} h'' = \Phi \\ h(0) = h_e < h_{\ell}^{aat}, \\ h(y^{aat, -}) = h_{\ell}^{aat} \end{cases}$$
in $]0, y^{aat}[$
$$\begin{cases} c_{p,l} D_e T' - \omega_{\ell} T'' = \Phi \\ T(0) = T_e < T^{aat}, \\ T(y^{aat, -}) = T^{aat}, \end{cases}$$
$$\begin{cases} D_e h' - \lambda_g h'' = \Phi \\ h(y^{aat, +}) = h_{\theta}^{aat} \\ \lim_{y \to \infty} h'(y) = \frac{\Phi}{D_e}, \end{cases}$$
in $]y^{aat}, +\infty[$
$$\begin{cases} c_{p,q} D_e T' - \omega_q T'' = \Phi \\ T(y^{aat, +}) = T^{aat}, \\ \lim_{y \to \infty} h'(y) = \frac{\Phi}{D_e}, \end{cases}$$
$$\lambda_g h'(y^{aat}) - \lambda_{\ell} h'(y^{aat}) = D_e(h_{\theta}^{aat} - h_{\ell}^{aat})$$
$$\omega_g T'(y^{aat, +}) - \omega_{\ell} T'(y^{aat, -}) = D_e(h_{\theta}^{aat} - h_{\ell}^{aat})$$
Remark: $[h(y^{aat})] = h_{\theta}^{aat} - h_{\ell}^{aat}$ Remark: $T(y^{aat, -}) = T(y^{aat, +}) = T^{aat}$

Link to the Stationary Stefan Problem - free boundary problem

Find the position $y^{\scriptscriptstyle{\rm Ault}}$ such that the over-determinate elliptic problem is satisfied:

Stationary Stefan problem on enthalpyStationary Stefan problem on temperature
$$\begin{cases} D_e h' - \lambda_{\ell} h'' = \Phi \\ h(0) = h_e < h_{\ell}^{\text{solt}}, & \text{in }]0, y^{\text{solt}} [\\ h(y^{\text{solt},-}) = h_{\ell}^{\text{solt}} & \text{in }]0, y^{\text{solt}} [\\ f(y^{\text{solt},-}) = h_{\ell}^{\text{solt}} & \text{in }]0, y^{\text{solt}} [\\ f(y^{\text{solt},+}) = h_{\vartheta}^{\text{solt}} & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} h'(y) = \frac{\Phi}{D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} h'(y) = \frac{\Phi}{D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} h'(y) = \frac{\Phi}{D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y) = \frac{\Phi}{c_{p,g}D_e}, & \text{in }]y^{\text{solt}}, +\infty [\\ \lim_{y \to \infty} T'(y^{\text{solt},+}) - \omega_{\ell}T'(y^{\text{solt},-}) = D_e(h_{\vartheta}^{\text{solt}} - h_{\ell}^{\text{solt}}) & \text{Remark: } T(y^{\text{solt},-}) = T(y^{\text{solt},+}) = T^{\text{solt}} & \text{Remark: }$$

Link to the Stationary Stefan Problem

Stefan Problem

- Sharp interface framework (no mixture allowed).
- By assumption, jump equals to $h_{g}^{\scriptscriptstyle{
 m adt}}-h_{\ell}^{\scriptscriptstyle{
 m adt}}.$

Our model

• Diffuse interface framework allowing for mixture.

• Jump equals to
$$\min\left\{rac{\lambda_{rak{d}}}{D_e}rac{\Phi}{D_e},h_{rak{d}}^{ ext{sat}}-h_{\ell}^{ ext{sat}}
ight\}$$

Same model when
$$rac{\lambda_{artheta}}{D_e} rac{\Phi}{D_e} \geq h_{artheta}^{ ext{soft}} - h_{arlepsilon}^{ ext{soft}}.$$

When our model involves a mixture, the Stefan Problem yields non-physical solution The last relation in the temperature formulation gives

$$\omega_{\mathfrak{f}}T'(y^{\mathrm{sal},+}) - \omega_{\mathfrak{f}}T'(y^{\mathrm{sal},-}) = D_{e}(h_{\mathfrak{f}}^{\mathrm{sal}} - h_{\mathfrak{f}}^{\mathrm{sal}}) \quad \rightsquigarrow \quad \omega_{\mathfrak{f}}T'(y^{\mathrm{sal},-}) = \lambda_{\mathfrak{f}}\frac{\Phi}{D_{e}} - D_{e}(h_{\mathfrak{f}}^{\mathrm{sal}} - h_{\mathfrak{f}}^{\mathrm{sal}}).$$

If $\frac{\Lambda_{\frac{\alpha}{2}}}{D_e} \frac{\Phi}{D_e} < h_{\frac{\alpha}{2}}^{\text{saft}} - h_{\ell}^{\text{saft}}$ (indicating the presence of a mixture in our model), we observe an implausible scenario near y^{saft} within the liquid phase: $T'(y^{\text{saft},-}) < 0$ and thus the temperature exceeds T^{saft} .

Link to the Stationary Stefan Problem

Stefan Problem

• Sharp interface framework (no mixture allowed).

Our model

• Diffuse interface framework allowing for mixture.

• Jump equals to
$$\min\left\{rac{\lambda_{rak{g}}}{D_e}rac{\Phi}{D_e},h_{rak{g}}^{ ext{sat}}-h_{\ell}^{ ext{sat}}
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Same model when
$$rac{\lambda_{artheta}}{D_e} rac{\Phi}{D_e} \geq h_{artheta}^{ ext{sat}} - h_{arlepsilon}^{ ext{sat}}.$$

When our model involves a mixture, the Stefan Problem yields non-physical solution The last relation in the temperature formulation gives

$$\omega_{\mathfrak{g}}T'(y^{\scriptscriptstyle \operatorname{sal},+}) - \omega_{\mathfrak{l}}T'(y^{\scriptscriptstyle \operatorname{sal},-}) = D_e(h_{\mathfrak{g}}^{\scriptscriptstyle \operatorname{sal}} - h_{\mathfrak{l}}^{\scriptscriptstyle \operatorname{sal}}) \quad \rightsquigarrow \quad \omega_{\mathfrak{l}}T'(y^{\scriptscriptstyle \operatorname{sal},-}) = \lambda_{\mathfrak{g}}\frac{\Phi}{D_e} - D_e(h_{\mathfrak{g}}^{\scriptscriptstyle \operatorname{sal}} - h_{\mathfrak{l}}^{\scriptscriptstyle \operatorname{sal}}).$$

If $\frac{\lambda_{\vartheta}}{D_e} \frac{\Phi}{D_e} < h_{\vartheta}^{\text{sat}} - h_{\ell}^{\text{sat}}$ (indicating the presence of a mixture in our model), we observe an implausible scenario near y^{sat} within the liquid phase: $T'(y^{\text{sat},-}) < 0$ and thus the temperature exceeds T^{sat} .

2. Steady-state model

- 2.1 The 1D steady-state model
- 2.2 Solution with $\lambda_{\ell} = \lambda_m = \lambda_{q_0} = 0$ (without diffusion)
- 2.3 Solution with $\lambda_{\ell}, \lambda_{m}, \lambda_{q} > 0$ (strictly positive diffusion)
- 2.4 Solution with $\lambda_m = 0$ and $\lambda_{\ell}, \lambda_q > 0$ (degenerate diffusion
- 2.5 Link to the Stationary Stefan Problem
- 2.6 Conclusion on the Steady-State Model

In the absence of diffusion, the mixture zone (= diffuse interface) always exists

Thermal diffusion in the gas phase reduces the mixture zone and can cause it to vanish if it is high enough:
 Neglect thermal diffusion in the gas phase if

$$rac{\lambda_{\mathfrak{f}}}{D_e}rac{\Phi}{D_e} \ll h_{\mathfrak{f}}^{\mathrm{ad}} - h_{\mathfrak{f}}^{\mathrm{ad}}$$

No mixture zone (= sharp interface) at steady state if

$$\frac{\lambda_{\mathfrak{f}}}{D_{\mathfrak{e}}}\frac{\Phi}{D_{\mathfrak{e}}} > h_{\mathfrak{f}}^{\mathrm{ant}} - h_{\mathfrak{f}}^{\mathrm{ant}}$$

--- Connection to the Stefan problem

cf. G.F., C. Galusinski, ESAIM: Proc. 2023, Vol. 72

Next step: Time-dependent model

- **()** In the absence of diffusion, the mixture zone (= diffuse interface) always exists
- O Thermal diffusion in the gas phase reduces the mixture zone and can cause it to vanish if it is high enough:
 - Neglect thermal diffusion in the gas phase if

$$\frac{\lambda_{\vartheta}}{D_e}\frac{\Phi}{D_e} \ll h_{\vartheta}^{\rm sat} - h_{\mathbb{l}}^{\rm sat}$$

• No mixture zone (= sharp interface) at steady state if

$$rac{\lambda_{artheta}}{D_e}rac{\Phi}{D_e} > h_{artheta}^{\mathrm{sat}} - h_{arlepsilon}^{\mathrm{sat}}$$

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cf. G.F., C. Galusinski, ESAIM: Proc. 2023, Vol. 72

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 \rightsquigarrow Connection to the Stefan problem

cf. G.F., C. Galusinski, ESAIM: Proc. 2023, Vol. 72

Next step: Time-dependent model

Remark: in a PWR, the ratio $\frac{\lambda_{\theta}}{D_e} \frac{\Phi}{D_e} / (h_{\theta}^{\text{solt}} - h_{\ell}^{\text{solt}})$ is approximately $1.3 \ 10^{-6} \rightsquigarrow$ the jump is negligible

3. Simplified Model

- 3.1 The simplified Equation of state
- 3.2 Numerical Time-Dependent Solution
- 3.3 Analytical Simplified Time-Dependent Solution
- 3.4 Conclusion about the Simplified Model

3. Simplified Model

3.1 The simplified Equation of state

3.2 Numerical Time-Dependent Solution3.3 Analytical Simplified Time-Dependent Solution3.4 Conclusion about the Simplified Model

The Simplified model

$$\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0\\ \partial_t (\varrho h) + \partial_y (\varrho h v) = \left[\Phi + \partial_y (\lambda(h) \partial_y h) \right]\\ \hline h \mapsto \varrho(h) \equiv 1 \text{ for all } h \end{cases}$$

$$\rightsquigarrow \qquad \begin{cases} \partial_y v = 0\\ \partial_t h + \partial_y (hv) = \left[\Phi + \partial_y \left(\lambda(h)\partial_y h\right)\right] \end{cases}$$

Simplified Model

$$\partial_t h + v \partial_y h - \partial_{yy}^2(L(h)) = \Phi$$
 in $\mathbb{R}^+ \times \mathbb{R}^+$

• L such that $L'(h) = \lambda(h)$, e.g.

$$L(h) \stackrel{\text{\tiny def}}{=} \begin{cases} \lambda_{\emptyset}(h - h_{\emptyset}^{\text{\tiny adl}}) & \text{if } h \leq h_{\emptyset}^{\text{\tiny adl}} \\ 0 & \text{if } h_{\emptyset}^{\text{\tiny adl}} < h < h_{\vartheta}^{\text{\tiny adl}} \\ \lambda_{\vartheta}(h - h_{\vartheta}^{\text{\tiny adl}}) & \text{if } h \geq h_{\vartheta}^{\text{\tiny adl}} \end{cases}$$

- $v(y) = v_e > 0$ constant
- $\Phi > 0$ constant

• BC:
$$h(y=0,t) = h_e < h_{l}^{\text{sol}}$$
, $\lim_{y \to \infty} h'(y) = \frac{\Phi}{De}$

• IC:
$$h(y, t = 0) = h_{init}(y) = h_e$$

The Simplified model

$$\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0\\ \partial_t (\varrho h) + \partial_y (\varrho h v) = \left[\Phi + \partial_y (\lambda(h) \partial_y h) \right]\\ \hline h \mapsto \varrho(h) \equiv 1 \text{ for all } h \end{cases}$$

$$\rightsquigarrow \qquad \begin{cases} \partial_y v = 0\\ \partial_t h + \partial_y (hv) = \left[\Phi + \partial_y \left(\lambda(h)\partial_y h\right)\right] \end{cases}$$

Simplified Model

$$\partial_t h + v \partial_y h - \partial_{yy}^2(L(h)) = \Phi \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+$$

• L such that $L'(h) = \lambda(h)$, e.g.

$$L(h) \stackrel{\mathrm{def}}{=} egin{cases} \lambda_{\ell}(h-h_{\ell}^{\mathrm{acl}}) & ext{if } h \leq h_{\ell}^{\mathrm{acl}} \ 0 & ext{if } h_{\ell}^{\mathrm{acl}} < h < h_{\mathfrak{g}}^{\mathrm{acl}} \ \lambda_{\mathfrak{g}}(h-h_{\mathfrak{g}}^{\mathrm{acl}}) & ext{if } h \geq h_{\mathfrak{g}}^{\mathrm{acl}} \end{cases}$$

- $v(y) = v_e > 0$ constant
- $\Phi > 0$ constant

• BC:
$$h(y=0,t) = h_e < h_{\ell}^{\text{soft}}, \lim_{y \to \infty} h'(y) = \frac{\Phi}{De}$$

• IC:
$$h(y,t=0) = h_{\text{init}}(y) = h_e$$

3. Simplified Model

3.1 The simplified Equation of state

3.2 Numerical Time-Dependent Solution

3.3 Analytical Simplified Time-Dependent Solution

3.4 Conclusion about the Simplified Model

Numerical Time-Dependent Solution

Front Tracking Method ~>> benchmark for evaluating the performance of other approaches

Explicit Scheme

$$rac{h^{n+1}-h^n}{\delta t}+v\partial_y h^n-\partial^2_{yy}(L(h^n))=\Phi \quad ext{in } \mathbb{R}^+$$

This is associated with a gradient scheme proposed by Eymard et al. in 2013.

→ the CFL condition 🐢 🐢

Implicit Scheme

$$\frac{h^{n+1}-h^n}{\delta t}+v\partial_y h^{n+1}-\partial_{yy}^2(L(h^{n+1}))=\Phi\quad\text{in }\mathbb{R}^+$$

This is associated with a gradient scheme as above.

 \rightsquigarrow Convergence of the fixed point when the jump is generated can be complicated 😅

Internship by L. Lamerand

Simulations

- No diffusion: Influence of Φ on Mixture Zone Width $\Delta x_{\kappa}^{\text{solt}}$ $\Phi = 2, \lambda_{\parallel} = 0, \lambda_{g} = 0$ $\Phi = 1, \lambda_{\parallel} = 0, \lambda_{g} = 0$
- **(a)** Diffusion in Liquid Phase: Influence on $y_{\emptyset}^{\text{sat}}$ and Slope at $y_{\emptyset}^{\text{sat}} \bullet \Phi = 1$, $\lambda_{\emptyset} = 0 \to 2$, $\lambda_{g} = 0$
- **③** Diffusion in Gas Phase: Influence on Mixture Zone width $\Delta y_{\kappa}^{\text{salt}}$ and jump at $y_{g}^{\text{salt}} \longrightarrow \Phi = 1$, $\lambda_{g} = 0$, $\lambda_{g} = 0 \rightarrow 1$
- Diffusion in Both Phases: Mixture Zone Existence $\Phi = 1, \lambda_0 = 0 \rightarrow 1, \lambda_0 = 1$
- **9** Diffusion in Both Phases: Mixture Zone Disappearance $\bullet \Phi = 1, \lambda_g = 1, \lambda_g = 1 \rightarrow 2$
- **(a)** Diffusion in Both Phases: No Mixture Zone, Focus on Slope ($\Phi = 1, \lambda_{\beta} = 1, \lambda_{\beta} = 2 \rightarrow 3$ ($\Phi = 1, \lambda_{\beta} = 1, \lambda_{\beta} = 1, \lambda_{\beta} = 3 \rightarrow 4$
- O Diffusion in Both Phases: No Mixture Zone, Jump in Liquid Phase $\Phi = \frac{1}{2}, \lambda_0 = 1, \lambda_0 = 5$ $\Phi = \frac{1}{2}, \lambda_0 = 0.1, \lambda_0 = 5$ $\Phi = \frac{1}{2}, \lambda_0 = 0, \lambda_0 = 5$

▲ Return

 $rac{\Phi}{v}=2$, $rac{\lambda_{\ell}}{v}=0$, $rac{\lambda_{\vartheta}}{v}=0$

$$\mathsf{Case}~\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{aot}}} - h_{\vartheta}^{_{\mathrm{aot}}}: \text{ mixture zone at steady state}$$



No diffusion: impact of varying Φ on $\Delta x_\kappa^{\rm sat}$, the width of the mixture zone.

 $rac{\Phi}{v}=2$, $rac{\lambda_{\ell}}{v}=0$, $rac{\lambda_{\vartheta}}{v}=0$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{act}}} - h_{\ell}^{_{\mathrm{act}}} : \, \mathsf{mixture \,\, zone \,\, at \,\, steady \,\, state}$$

No diffusion: impact of varying Φ on $\Delta x_{\kappa}^{{}_{\rm Adl}}$, the width of the mixture zone.

◀ Return

 $rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=0$, $rac{\lambda_{g}}{v}=0$

$$\mathsf{Case}~\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{aat}}} - h_{\vartheta}^{_{\mathrm{aat}}}: \text{ mixture zone at steady state}$$



 $rac{\Phi}{v}=1$, $rac{\lambda_{arrho}}{v}=0$, $rac{\lambda_{arrho}}{v}=0$

$$\mathsf{Case}\,\,\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{act}}} - h_{\ell}^{_{\mathrm{act}}}:\,\mathsf{mixture}\,\,\mathsf{zone}\,\,\mathsf{at}\,\,\mathsf{steady}\,\,\mathsf{state}$$

No diffusion: influence of Φ on $\Delta x_{\kappa}^{\rm salt}$ the width of the mixture zone

 $rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=2$, $rac{\lambda_{g}}{v}=0$

$$\mathsf{Case}~\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{act}}} - h_{\varrho}^{_{\mathrm{act}}}: \text{ mixture zone at steady state}$$



Diffusion Liquid Phase: effect of liquid phase diffusion on y_{ℓ}^{sal} and its associated slope.

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 $rac{\Phi}{v}=1$, $rac{\lambda_{arrho}}{v}=2$, $rac{\lambda_{arrho}}{v}=0$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{act}}} - h_{\ell}^{_{\mathrm{act}}} : \, \mathsf{mixture \,\, zone \,\, at \,\, steady \,\, state}$$

Diffusion Liquid Phase: effect of liquid phase diffusion on $y_{\ell}^{_{aat}}$ and its associated slope.

◀ Return

61/92

 $rac{\Phi}{v}=1$, $rac{\lambda_{arrho}}{v}=0$, $rac{\lambda_{arrho}}{v}=1$





Diffusion in Gas Phase: influence of gas phase diffusion on $\Delta y_{\kappa}^{\text{sat}}$, the width of the mixture zone, and the jump at y_{θ}^{sat} .

 $rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=0$, $rac{\lambda_{g}}{v}=1$

$$\mathsf{Case}\,\,\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{aat}}} - h_{\ell}^{_{\mathrm{aat}}}:\,\mathsf{mixture}\,\,\mathsf{zone}\,\,\mathsf{at}\,\,\mathsf{steady}\,\,\mathsf{state}$$

Diffusion in Gas Phase: influence of gas phase diffusion on $\Delta y_{\kappa}^{\text{sat}}$, the width of the mixture zone, and the jump at y_{g}^{sat} .

 $rac{\Phi}{v}=1$, $rac{\lambda_{arrho}}{v}=1$, $rac{\lambda_{arrho}}{v}=1$

$$\mathsf{Case}~\frac{\lambda_\vartheta}{v}\frac{\Phi}{v} < h_\vartheta^{_{\mathrm{aat}}} - h_\vartheta^{_{\mathrm{aat}}}: \text{ mixture zone at steady state}$$



Diffusion in Both Phases: impact on the existence of the mixture zone.

 $rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=1$, $rac{\lambda_{\vartheta}}{v}=1$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} < h_{\vartheta}^{_{\mathrm{act}}} - h_{\varrho}^{_{\mathrm{act}}} : \, \mathsf{mixture \,\, zone \,\, at \,\, steady \,\, state}$$

Diffusion in Both Phases: impact on the existence of the mixture zone.


$rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=1$, $rac{\lambda_{artheta}}{v}=2$

$$\mathsf{Case}\ \frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v}\geq h_{\vartheta}^{_{\mathrm{act}}}-h_{\varrho}^{_{\mathrm{act}}}: \text{ no mixture zone at steady state}$$



Diffusion in Both Phases: disappearance of the mixture zone.

 $rac{\Phi}{v}=1$, $rac{\lambda_{artheta}}{v}=1$, $rac{\lambda_{artheta}}{v}=2$

$$\mathsf{Case}~\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v} \geq h_{\vartheta}^{_{\mathrm{act}}} - h_{l}^{_{\mathrm{act}}} \text{ : no mixture zone at steady state}$$

Diffusion in Both Phases: disappearance of the mixture zone.



 $rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=1$, $rac{\lambda_{g}}{v}=3$

$$\mathsf{Case}\ \frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v}\geq h_{\vartheta}^{_{\mathrm{act}}}-h_{\varrho}^{_{\mathrm{act}}}: \text{ no mixture zone at steady state}$$



 $rac{\Phi}{v}=1$, $rac{\lambda_{artheta}}{v}=1$, $rac{\lambda_{artheta}}{v}=3$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} \geq h_{\vartheta}^{_{\mathrm{act}}} - h_{\ell}^{_{\mathrm{act}}}: \, \mathsf{no} \,\, \mathsf{mixture} \,\, \mathsf{zone} \,\, \mathsf{at} \,\, \mathsf{steady} \,\, \mathsf{state}$$

Diffusion in Both Phases – no mixture zone: focus on the slope at y_{l}^{saft} .

 $rac{\Phi}{v}=1$, $rac{\lambda_{\ell}}{v}=1$, $rac{\lambda_{g}}{v}=4$

$$\mathsf{Case}~\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v}\geq h_{\vartheta}^{\scriptscriptstyle\mathrm{aut}}-h_{\ell}^{\scriptscriptstyle\mathrm{aut}}~\text{: no mixture zone at steady state}$$



Diffusion in Both Phases – no mixture zone: focus on the slope at y_0^{soft} that can be $> \frac{\Phi}{v}$

Return

 $rac{\Phi}{v}=1$, $rac{\lambda_{artheta}}{v}=1$, $rac{\lambda_{artheta}}{v}=4$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} \geq h_{\vartheta}^{_{\mathrm{act}}} - h_{\ell}^{_{\mathrm{act}}}: \, \mathsf{no} \,\, \mathsf{mixture} \,\, \mathsf{zone} \,\, \mathsf{at} \,\, \mathsf{steady} \,\, \mathsf{state}$$

Diffusion in Both Phases – no mixture zone: focus on the slope at $y_l^{
m soft}$ that can be $>rac{\Phi}{v}$



 $rac{\Phi}{v}=rac{1}{2}$, $rac{\lambda_{arrho}}{v}=1$, $rac{\lambda_{arrho}}{v}=5$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} \geq h_{\vartheta}^{_{\mathrm{aut}}} - h_{\varrho}^{_{\mathrm{aut}}} : \, \mathsf{no} \,\, \mathsf{mixture} \,\, \mathsf{zone} \,\, \mathsf{at} \,\, \mathsf{steady} \,\, \mathsf{state}$$



 $rac{\Phi}{v}=rac{1}{2}$, $rac{\lambda_{\ell}}{v}=1$, $rac{\lambda_{artheta}}{v}=5$

$$\mathsf{Case}\,\,\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v}\geq h_{\vartheta}^{_{\mathrm{act}}}-h_{l}^{_{\mathrm{act}}}:\,\mathsf{no}\,\,\mathsf{mixture}\,\,\mathsf{zone}\,\,\mathsf{at}\,\,\mathsf{steady}\,\,\mathsf{state}$$

Diffusion in Both Phases – no mixture zone: focus on the slope at $y^{\scriptscriptstyle{\rm dot},-}$

 $rac{\Phi}{v}=rac{1}{2}$, $rac{\lambda_{\ell}}{v}=0.1$, $rac{\lambda_{artheta}}{v}=5$

$$\mathsf{Case}\ \frac{\lambda_\vartheta}{v}\frac{\Phi}{v}\geq h_\vartheta^{_{\mathrm{adt}}}-h_{\ell}^{_{\mathrm{adt}}}: \text{ no mixture zone at steady state}$$



Diffusion in Both Phases – no mixture zone: focus on the slope at $y^{_{\rm Adl},-}$ when $\lambda_{\emptyset} o 0$

 $\frac{\Phi}{v}=rac{1}{2}$, $rac{\lambda_{\ell}}{v}=0.1$, $rac{\lambda_{\vartheta}}{v}=5$

$$\mathsf{Case} \,\, \frac{\lambda_{\vartheta}}{v} \frac{\Phi}{v} \geq h_{\vartheta}^{_{\mathrm{act}}} - h_{\ell}^{_{\mathrm{act}}} : \, \mathsf{no} \,\, \mathsf{mixture} \,\, \mathsf{zone} \,\, \mathsf{at} \,\, \mathsf{steady} \,\, \mathsf{state}$$

Diffusion in Both Phases – no mixture zone: focus on the slope at $y^{_{\rm solt},-}$ when $\lambda_{\ell} o 0$



 $\frac{\Phi}{v}=\frac{1}{2}$, $\frac{\lambda_{\ell}}{v}=0$, $\frac{\lambda_{g}}{v}=5$

$$\mathsf{Case}\ \frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v}\geq h_{\vartheta}^{_{\mathrm{act}}}-h_{\varrho}^{_{\mathrm{act}}}: \text{ no mixture zone at steady state}$$



Diffusion only in the Gas – no mixture zone: jump in the liquid phase at y^{soft}

 $\frac{\Phi}{v}=\frac{1}{2}$, $\frac{\lambda_{\ell}}{v}=0$, $\frac{\lambda_{\theta}}{v}=5$

$$\mathsf{Case}\,\,\frac{\lambda_{\vartheta}}{v}\frac{\Phi}{v}\geq h_{\vartheta}^{_{\mathrm{act}}}-h_{l}^{_{\mathrm{act}}}:\,\mathsf{no}\,\,\mathsf{mixture}\,\,\mathsf{zone}\,\,\mathsf{at}\,\,\mathsf{steady}\,\,\mathsf{state}$$

Diffusion only in the Gas – no mixture zone: jump in the liquid phase at $y^{\scriptscriptstyle{
m act}}$

3. Simplified Model

- 3.1 The simplified Equation of state
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Validation?

- Asymptotic behavior? V
- Transient behavior?
 - Comparison with an interface tracking code? \checkmark
 - Comparison with an exact solution (in a simplified case) ?

Focusing on the jump displacement:

- Consider thermal diffusion only in the gas $(\lambda_{\ell}=\lambda_{\scriptscriptstyle m}=0)$
- Seek an exact transient solution in the form of a traveling wave:
 - ullet Interface displacement imposed: $y_{g}^{ ext{sat}}(t)=y_{g}^{ ext{sat}}(0)-ct$
 - Amplitude of the jump imposed: $h_a^{
 m sat} h_m^{
 m sat}$ for all t
 - If $h_e(t)=h_m^{
 m sal}-\eta y_g^{
 m sal}(t)$, the solution (which does not exhibit asymptotic behavior) is given by

$$h(t,y) = \begin{cases} h_m^{\mathrm{salt}} + \eta(y - y_{\vartheta}^{\mathrm{salt}}(t)) & \text{if } y < y_{\vartheta}^{\mathrm{salt}}(t) \\ h_{\vartheta}^{\mathrm{salt}} + \eta(y - y_{\vartheta}^{\mathrm{salt}}(t)) & \text{if } y > y_{\vartheta}^{\mathrm{salt}}(t) \end{cases} \qquad \eta \stackrel{\mathrm{def}}{=} \sqrt{\frac{h_{\vartheta}^{\mathrm{salt}} - h_m^{\mathrm{salt}}}{\lambda_g/\Phi}}$$

Validation?

- Asymptotic behavior? V
- Transient behavior?
 - Comparison with an interface tracking code? \checkmark
 - Comparison with an exact solution (in a simplified case) ?

Focusing on the jump displacement:

- Consider thermal diffusion only in the gas $(\lambda_{\ell}=\lambda_{m}=0)$
- Seek an exact transient solution in the form of a traveling wave:
 - Interface displacement imposed: $y_{\mathfrak{A}}^{\mathrm{sat}}(t)=y_{\mathfrak{A}}^{\mathrm{sat}}(0)-ct$
 - Amplitude of the jump imposed: $h_{a}^{\mathrm{sat}} h_{m}^{\mathrm{sat}}$ for all t
 - If $h_e(t)=h_m^{
 m sat}-\eta y_{eta}^{
 m sat}(t)$, the solution (which does not exhibit asymptotic behavior) is given by

$$h(t,y) = \begin{cases} h_m^{\mathrm{saf}} + \eta(y - y_{\vartheta}^{\mathrm{saf}}(t)) & \text{if } y < y_{\vartheta}^{\mathrm{saf}}(t) \\ h_{\vartheta}^{\mathrm{saf}} + \eta(y - y_{\vartheta}^{\mathrm{saf}}(t)) & \text{if } y > y_{\vartheta}^{\mathrm{saf}}(t) \end{cases} \qquad \eta \stackrel{\mathrm{def}}{=} \sqrt{\frac{h_{\vartheta}^{\mathrm{saf}} - h_m^{\mathrm{saf}}}{\lambda_{\vartheta}/\Phi}}$$

Validation through Traveling Wave Solution

3. Simplified Model

3.1 The simplified Equation of state
3.2 Numerical Time-Dependent Solution
3.3 Analytical Simplified Time-Dependent Solution
3.4 Conclusion about the Simplified Model

() Numerical schemes for phase appearance/disappearance and jumps are robust

- On The simulations exhibit precise asymptotic behavior
- The simulations demonstrate plausible transient behavior
- Validation of time-dependent jump displacement using analytical solution as traveling waves

Next step: Density $arrho(h)
ot\equiv 1 \rightsquigarrow$ complete LMNC model

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Validation of time-dependent jump displacement using analytical solution as traveling waves

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- **0** Numerical schemes for phase appearance/disappearance and jumps are robust
- On the simulations exhibit precise asymptotic behavior
- The simulations demonstrate plausible transient behavior
- Validation of time-dependent jump displacement using analytical solution as traveling waves

Next step: Density $\varrho(h) \not\equiv 1 \rightsquigarrow$ complete LMNC model

4. Full Model

- 4.1 The "real" Equation of state
- 4.2 The Full Model with Degenerating Diffusion
- 4.3 Concluding Remarks on the Full Model

4. Full Model

4.1 The "real" Equation of state

4.2 The Full Model with Degenerating Diffusion

4.3 Concluding Remarks on the Full Model

The "real" Equation of state

• ϱ is piecewise defined w.r.t. $h_{\kappa}^{\rm salt}(p)$

$$\varrho(h,p) = \begin{cases} \varrho_{\mathbb{I}}(h,p), & \text{if } h \leq h_{\mathbb{I}}^{\text{soft}}(p) \\ \varrho_{\text{m}}(h,p), & \text{if } h_{\mathbb{I}}^{\text{soft}}(p) < h < h_{\mathfrak{H}}^{\text{soft}}(p) \\ \varrho_{\mathfrak{H}}(h,p), & \text{if } h \geq h_{\mathfrak{H}}^{\text{soft}}(p) \end{cases} \quad \text{e.g. if SG} \quad \varrho_{\kappa}(h,p) = \frac{\zeta_{\kappa}(p)}{h - q_{\kappa}}$$

• At constant pressure $p = p_*$



The "real" Equation of state

• ϱ is piecewise defined w.r.t. $h_{\kappa}^{\text{salt}}(p)$

$$\varrho(h,p) = \begin{cases} \varrho_{\mathbb{I}}(h,p), & \text{if } h \leq h_{\mathbb{I}}^{\text{soft}}(p) \\ \varrho_{\text{m}}(h,p), & \text{if } h_{\mathbb{I}}^{\text{soft}}(p) < h < h_{\mathfrak{H}}^{\text{soft}}(p) \\ \varrho_{\mathfrak{H}}(h,p), & \text{if } h \geq h_{\mathfrak{H}}^{\text{soft}}(p) \end{cases} \quad \text{e.g. if SG} \quad \varrho_{\kappa}(h,p) = \frac{\zeta_{\kappa}(p)}{h - q_{\kappa}}$$

• At constant pressure $p = p_*$



4. Full Model

4.2 The Full Model with Degenerating Diffusion ${\bullet}$ Challenges when $\lambda_{m}=0$

- 2 Predictor-Corrector Approach
- 3 Relaxation Approach

The Full Model with Degenerating Diffusion

When $\lambda_{m}=0$, the steady solution exhibits...

a jump in
$$h \, \rightsquigarrow \,$$
 a jump in $\varrho(h) \, \rightsquigarrow \,$ a jump in $v = \frac{D_e}{\varrho(h)}$

We cannot work with the non-conservative form as in the simplified model

$$\begin{cases} \partial_t \varrho + \partial_y(\varrho v) = 0\\ \partial_t(\varrho h) + \partial_y(\varrho h v) - \partial_{yy} L(h) = \Phi \end{cases}$$

The Full Model with Degenerating Diffusion

When $\lambda_{\rm m}=0$, the steady solution exhibits...

a jump in
$$h \ \rightsquigarrow$$
 a jump in $\varrho(h) \ \rightsquigarrow$ a jump in $v = \frac{D_e}{\varrho(h)}$

 \bigvee We cannot work with the non-conservative form as in the simplified model

∜

$$\begin{cases} \partial_t \varrho + \partial_y(\varrho v) = 0\\ \\ \partial_t(\varrho h) + \partial_y(\varrho h v) - \partial_{yy}L(h) = \Phi \end{cases}$$

Numerical schemes?

How can we compute h^{n+1} and v^{n+1} from h^n and v^n ?

Direct Explicit schemes are \quad as there is no explicit time derivative of v (the system is not hyperbolic):

$$\begin{cases} \partial_t \varrho_i + \partial_y (\varrho v)^n = 0 \\ & \longrightarrow h^{n+1} (\text{overdetermined}), \text{ but not } v^{n+1} \\ \partial_t (\varrho h)_i + \partial_y (\varrho h v)^n - \partial_{yy} L(h^n) = \Phi \end{cases}$$

1. Eo

Degen

3. Conclusion

Numerical schemes?

How can we compute h^{n+1} and v^{n+1} from h^n and v^n ?

Direct Explicit schemes are \bigotimes as there is no explicit time derivative of v (the system is not hyperbolic):

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79/92

1. Eos

Numerical schemes?



Ne consider three approaches:

An Implicit Scheme:

 $\begin{aligned} \left(\partial_t \varrho_i + \partial_y (\varrho v)^{n+1} &= 0 \\ \\ \partial_i (\varrho h)_i + \partial_y (\varrho h v)^{n+1} - \partial_{yy} L(h^{n+1}) &= 0 \end{aligned} \right) \\ \end{aligned}$

ut what about convergence?

(≃ one step of a fixed point for an implicit scheme)

A Relaxation Scheme

1. EoS

Numerical schemes?



We consider three approaches:

An Implicit Scheme:

$$\begin{cases} \partial_t \varrho_i + \partial_y (\varrho v)^{n+1} = 0 & \\ \partial_t (\varrho h)_i + \partial_y (\varrho h v)^{n+1} - \partial_{yy} L(h^{n+1}) = \Phi & \end{cases} \xrightarrow{\sim} h^{n+1} \text{ and } v^{n+1} \end{cases}$$

but what about convergence? 😅

- A Predictor-Corrector Scheme (≃ one step of a fixed point for an implicit scheme)
- A Relaxation Scheme

1. EoS

Numerical schemes?



We consider three approaches:

An Implicit Scheme:

 $\begin{cases} \partial_t \varrho_i + \partial_y (\varrho v)^{n+1} = 0 & \\ \partial_t (\varrho h)_i + \partial_y (\varrho h v)^{n+1} - \partial_{yy} L(h^{n+1}) = \Phi & \end{cases} \xrightarrow{\sim} h^{n+1} \text{ and } v^{n+1} \end{cases}$

but what about convergence? 😅

A Predictor-Corrector Scheme
 (≃ one step of a fixed point for an implicit scheme)

A Relaxation Scheme

1. EoS

Numerical schemes?



We consider three approaches:

An Implicit Scheme:

 $\begin{cases} \partial_t \varrho_i + \partial_y (\varrho v)^{n+1} = 0 & \\ & \longrightarrow h^{n+1} \text{ and } v^{n+1} \\ & \partial_t (\varrho h)_i + \partial_y (\varrho h v)^{n+1} - \partial_{yy} L(h^{n+1}) = \Phi & \end{cases}$

but what about convergence? 😅

- A Predictor-Corrector Scheme (~ one step of a fixed point for an implicit scheme)
- A Relaxation Scheme

2 Predictor-Corrector Approach

Predictor step:

$$\begin{cases} \partial_t \varrho_i + \partial_y (\varrho v)^n = 0 \\ & \longrightarrow h^{n+1} \text{or } \varrho^{n+1} \text{ (overdetermined)} \\ \partial_t (\varrho h)_i + \partial_y (\varrho h v)^n - \partial_{yy} L(h^n) = \Phi \end{cases}$$

Orrector step:

$$\begin{cases} \partial_t \varrho_i + \partial_y (\varrho v)^{n+1} = 0 \\ & \rightsquigarrow v^{n+1} \text{ (overdetermined)} \\ \partial_t (\varrho h)_i + \partial_y (\varrho h v)^{n+1} - \partial_{yy} L(h^{n+1}) = \Phi \end{cases}$$
2 Predictor-Corrector Approach

2 Predictor-Corrector Approach – Validation?

- Asymptotic behavior?
- Transient behavior?
 - Plausible? 🗸
 - Conservation at each instant? Partially verified
 - Comparison with an exact solution (in a simplified case) ?

Focus on an exact transient solution in the form of a traveling wave:

- Focusing on the jump displacement (thermal diffusion only in the gas)
- Interface displacement imposed: $y_{g}^{\mathrm{sat}}(t) = y_{g}^{\mathrm{sat}}(0) ct$
- Amplitude of the jump imposed: $h_{g}^{
 m sat} h_{m}^{
 m sat}$ for all t
- If $h_e(t) = h_m^{\text{solt}} \eta y_{\frac{\alpha}{\beta}}^{\text{solt}}(t)$, the solution (which does not exhibit asymptotic behavior) is given by h and η as in the simplified model and v as a function of h:

$$v(h) = \begin{cases} c + \frac{\Phi}{\eta \varrho(h_m^s)} + \frac{\Phi}{\eta \zeta_m} (h - h_m^{\text{sal}}(t)) & \text{if } h < h_m^{\text{sal}}(t) \\ c + \frac{\Phi}{\eta \varrho(h_g^s)} + \frac{\Phi}{\eta \zeta_g} (h - h_g^{\text{sal}}(t)) & \text{if } h > h_g^{\text{sal}}(t) \end{cases} \qquad \qquad v_e(t) = v(h_e(t))$$

⁽²⁾ Predictor-Corrector Approach – Validation?

- Asymptotic behavior?
- Transient behavior?
 - Plausible? 🗸
 - Conservation at each instant? Partially verified
 - Comparison with an exact solution (in a simplified case) ?

Focus on an exact transient solution in the form of a traveling wave:

- Focusing on the jump displacement (thermal diffusion only in the gas)
- Interface displacement imposed: $y_{\rm g}^{\rm soft}(t)=y_{\rm g}^{\rm soft}(0)-ct$
- \bullet Amplitude of the jump imposed: $h_g^{\rm sat}-h_m^{\rm sat}$ for all t
- If $h_e(t) = h_m^{\text{solt}} \eta y_{\theta}^{\text{solt}}(t)$, the solution (which does not exhibit asymptotic behavior) is given by h and η as in the simplified model and v as a function of h:

$$v(h) = \begin{cases} c + \frac{\Phi}{\eta_{\ell}(h_m^s)} + \frac{\Phi}{\eta\zeta_m}(h - h_m^{\text{ad}^{\text{t}}}(t)) & \text{if } h < h_m^{\text{ad}^{\text{t}}}(t) \\ c + \frac{\Phi}{\eta_{\ell}(h_g^s)} + \frac{\Phi}{\eta\zeta_g}(h - h_g^{\text{ad}^{\text{t}}}(t)) & \text{if } h > h_g^{\text{ad}^{\text{t}}}(t) \end{cases} \qquad \qquad v_e(t) = v(h_e(t))$$

2 Predictor-Corrector Approach

1. Eos

2. Degenerat

③ Relaxation Approach

$4\text{-}\mathrm{LMNC}$

$$\begin{aligned} &\left\{ \begin{aligned} \partial_{\varepsilon} \varrho_{\varepsilon} + \partial_{y} (\varrho_{\varepsilon} v_{\varepsilon}) &= 0 \\ &\partial_{\varepsilon} (\varrho_{\varepsilon} h_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} h_{\varepsilon} v_{\varepsilon}) &= \Phi + \partial_{y} (\omega_{\varepsilon} \partial_{y} T_{\varepsilon}) \\ &\partial_{\varepsilon} (\varrho_{\varepsilon} \varphi_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} \varphi_{\varepsilon} v_{\varepsilon}) &= \frac{1}{\varepsilon} \varrho_{\varepsilon} \left(\varphi^{\text{sub}}(h_{\varepsilon}) - \varphi_{\varepsilon} \right) \end{aligned} \end{aligned}$$

- Unknowns: h_{ε} , v_{ε} , φ_{ε} (mass fraction)
- EoS iso-Tp: $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto \varrho_{\varepsilon}$ and $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto T_{\varepsilon}$
- Diffusion:

$$\omega_{\mathbf{z}}\partial_{\mathbf{y}}T_{\mathbf{z}} = \frac{\omega_{\mathbf{z}}}{c_{\mathbf{y}}(\varphi_{\mathbf{z}})}\partial_{\mathbf{y}}h_{\mathbf{z}} + \omega_{\mathbf{z}}\frac{\partial T_{\mathbf{z}}}{\partial\varphi}\partial_{\mathbf{y}}\varphi$$



(3)-LMNC

$$\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0\\ \partial_t (\varrho h) + \partial_y (\varrho h v) = \Phi + \partial_y (\omega \partial_y T) \end{cases}$$

• Unknowns: h, v• EoS iso- $Tp\mu$: $h \mapsto \rho$ and $h \mapsto T \rightsquigarrow$ mass fraction



Diffusion:

$$\omega \partial_y T = \begin{cases} \lambda_{\mathbb{I}} \partial_y h & \text{if } h \leq h_{\mathbb{I}}^{\text{sat}} \\ \mathbf{0} & \text{if } h_{\mathbb{I}}^{\text{sat}} < h < h_g^{\text{sat}} \\ \lambda_g \partial_y h & \text{if } h \geq h_g^{\text{sat}} \end{cases}$$

• Since $\varrho_{\varepsilon}(h, \varphi^{\text{sat}}(h)) = \varrho(h)$ and $T_{\varepsilon}(h, \varphi^{\text{sat}}(h)) = T^{\text{sat}}$, formally 4-LMNC \longrightarrow (3)-LMNC

• In the 4-LMNC model the phase is not identified by the value of h_{e} but $arphi_{e}$: we can have a mixture even if $h_{e} > h_{a}^{ad}$

③ Relaxation Approach

$4\text{-}\mathrm{LMNC}$

$$\begin{cases} \partial_{\varepsilon} \varrho_{\varepsilon} + \partial_{y} (\varrho_{\varepsilon} v_{\varepsilon}) = 0\\ \partial_{\varepsilon} (\varrho_{\varepsilon} h_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} h_{\varepsilon} v_{\varepsilon}) = \Phi + \partial_{y} (\omega_{\varepsilon} \partial_{y} T_{\varepsilon})\\ \partial_{\varepsilon} (\varrho_{\varepsilon} \varphi_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} \varphi_{\varepsilon} v_{\varepsilon}) = \frac{1}{\varepsilon} \varrho_{\varepsilon} \left(\varphi^{\text{sol}}(h_{\varepsilon}) - \varphi_{\varepsilon} \right) \end{cases}$$

- Unknowns: h_{ε} , v_{ε} , φ_{ε} (mass fraction)
- EoS iso-Tp: $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto \varrho_{\varepsilon}$ and $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto T_{\varepsilon}$
- Diffusion:

$$\omega_{\mathbf{z}}\partial_{\mathbf{y}}T_{\mathbf{z}} = \frac{\omega_{\mathbf{z}}}{c_{\mathbf{y}}(\varphi_{\mathbf{z}})}\partial_{\mathbf{y}}h_{\mathbf{z}} + \omega_{\mathbf{z}}\frac{\partial T_{\mathbf{z}}}{\partial\varphi}\partial_{\mathbf{y}}\varphi$$



(3)-Lmnc

$$\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0\\ \partial_t (\varrho h) + \partial_y (\varrho h v) = \Phi + \partial_y (\omega \partial_y T) \end{cases}$$

- Unknowns: h, v
- \bullet EoS iso- $Tp\mu :$ $h\mapsto \varrho$ and $h\mapsto T$ \leadsto mass fraction

$$arphi^{\mathrm{sat}}(h) \stackrel{\mathrm{def}}{=} egin{cases} 0 & \mathrm{if} \ h \leq h_{\ell}^{\mathrm{sat}} \ rac{h-h_{\ell}^{\mathrm{sat}}}{h_{g}^{\mathrm{sat}}-h_{\ell}^{\mathrm{sat}}} & \mathrm{if} \ h_{\ell}^{\mathrm{sat}} < h < h_{g}^{\mathrm{sat}} \ 1 & \mathrm{if} \ h \geq h_{g}^{\mathrm{sat}} \end{cases}$$

Diffusion:

$$\omega \partial_y T = \begin{cases} \lambda_{\mathbb{I}} \partial_y h & \text{if } h \leq h_{\mathbb{I}}^{\text{salt}} \\ \mathbf{0} & \text{if } h_{\mathbb{I}}^{\text{salt}} < h < h_g^{\text{salt}} \\ \lambda_g \partial_y h & \text{if } h \geq h_g^{\text{salt}} \end{cases}$$

• Since $\varrho_{\varepsilon}(h, \varphi^{\text{sat}}(h)) = \varrho(h)$ and $T_{\varepsilon}(h, \varphi^{\text{sat}}(h)) = T^{\text{sat}}$, formally 4-LMNC \longrightarrow (3)-LMNC

• In the 4-LMNC model the phase is not identified by the value of $h_{arepsilon}$ but $arphi_{arepsilon}$: we can have a mixture even if $h_{arepsilon} > h_{ ext{a}}^{ ext{ad}}$

③ Relaxation Approach

$4\text{-}\mathrm{Lmnc}$

$$\begin{cases} \partial_{t} \varrho_{\varepsilon} + \partial_{y} (\varrho_{\varepsilon} v_{\varepsilon}) = 0\\ \partial_{t} (\varrho_{\varepsilon} h_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} h_{\varepsilon} v_{\varepsilon}) = \Phi + \partial_{y} (\omega_{\varepsilon} \partial_{y} T_{\varepsilon})\\ \partial_{t} (\varrho_{\varepsilon} \varphi_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} \varphi_{\varepsilon} v_{\varepsilon}) = \frac{1}{\varepsilon} \varrho_{\varepsilon} \left(\varphi^{\text{out}}(h_{\varepsilon}) - \varphi_{\varepsilon} \right) \end{cases}$$

Unknowns: h_ε, v_ε, φ_ε (mass fraction)
EoS iso-Tp: (h_ε, φ_ε) → g_ε and (h_ε, φ_ε) → T_ε
Diffusion:

$$\omega_{\varepsilon}\partial_{y}T_{\varepsilon} = \frac{\omega_{\varepsilon}}{c_{p}(\varphi_{\varepsilon})}\partial_{y}h_{\varepsilon} + \omega_{\varepsilon}\frac{\partial T_{\varepsilon}}{\partial\varphi}\partial_{y}\varphi$$



 $\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0\\ \partial_t (\varrho h) + \partial_y (\varrho h v) = \Phi + \partial_y (\omega \partial_y T) \end{cases}$

(3)-LMNC

- Unknowns: h, v
- \bullet EoS iso- $Tp\mu :$ $h\mapsto \varrho$ and $h\mapsto T$ \leadsto mass fraction

$$arphi^{ ext{solt}}(h) \stackrel{ ext{def}}{=} egin{cases} 0 & ext{if } h \leq h_{\ell}^{ ext{solt}} \ rac{h - h_{\ell}^{ ext{solt}}}{h_{\ell}^{ ext{solt}} - h_{\ell}^{ ext{solt}}} & ext{if } h_{\ell}^{ ext{solt}} < h < h_{g}^{ ext{solt}} \ 1 & ext{if } h \geq h_{g}^{ ext{solt}} \end{cases}$$

Diffusion:

$$\omega \partial_y T = \begin{cases} \lambda_{\mathbb{I}} \partial_y h & \text{if } h \leq h_{\mathbb{I}}^{\text{ad}} \\ \mathbf{0} & \text{if } h_{\mathbb{I}}^{\text{ad}} < h < h_{\mathfrak{g}}^{\text{ad}} \\ \lambda_{\mathfrak{g}} \partial_y h & \text{if } h \geq h_{\mathfrak{g}}^{\text{ad}} \end{cases}$$

• Since $\varrho_{\varepsilon}(h, \varphi^{\text{sol}}(h)) = \varrho(h)$ and $T_{\varepsilon}(h, \varphi^{\text{sol}}(h)) = T^{\text{sol}}$, formally 4-LMNC \longrightarrow (3)-LMNC

• In the 4-LMNC model the phase is not identified by the value of $h_{arepsilon}$ but $arphi_arepsilon$: we can have a mixture even if $h_arepsilon > h_{ ext{a}}^{ ext{ad}}$

③ Relaxation Approach

$\text{4-}\mathrm{Lmnc}$

$$\begin{cases} \partial_{t} \varrho_{\varepsilon} + \partial_{y} (\varrho_{\varepsilon} v_{\varepsilon}) = 0\\ \partial_{t} (\varrho_{\varepsilon} h_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} h_{\varepsilon} v_{\varepsilon}) = \Phi + \partial_{y} (\omega_{\varepsilon} \partial_{y} T_{\varepsilon})\\ \partial_{t} (\varrho_{\varepsilon} \varphi_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} \varphi_{\varepsilon} v_{\varepsilon}) = \frac{1}{\varepsilon} \varrho_{\varepsilon} \left(\varphi^{\text{solt}} (h_{\varepsilon}) - \varphi_{\varepsilon} \right) \end{cases}$$

- Unknowns: h_{ε} , v_{ε} , φ_{ε} (mass fraction)
- EoS iso-Tp: $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto \varrho_{\varepsilon}$ and $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto T_{\varepsilon}$
- Diffusion:

$$\omega_{\varepsilon}\partial_{y}T_{\varepsilon} = \frac{\omega_{\varepsilon}}{c_{p}(\varphi_{\varepsilon})}\partial_{y}h_{\varepsilon} + \omega_{\varepsilon}\frac{\partial T_{\varepsilon}}{\partial\varphi}\partial_{y}\varphi_{\varepsilon}$$



(3)-LMNC $\int \partial_t \varrho + \partial_y(\varrho v) = 0$

$$\left\{\partial_t(\varrho h) + \partial_y(\varrho h v) = \Phi + \partial_y(\omega \partial_y T)\right\}$$

- Unknowns: h, v
- EoS iso- $Tp\mu$: $h\mapsto \varrho$ and $h\mapsto T\rightsquigarrow$ mass fraction

$$arphi^{ ext{solt}}(h) \stackrel{ ext{def}}{=} egin{cases} 0 & ext{if } h \leq h_{\ell}^{ ext{solt}} \ rac{h - h_{\ell}^{ ext{solt}}}{h_{\ell}^{ ext{solt}} - h_{\ell}^{ ext{solt}}} & ext{if } h_{\ell}^{ ext{solt}} < h < h_{g}^{ ext{solt}} \ 1 & ext{if } h \geq h_{g}^{ ext{solt}} \end{cases}$$

Diffusion:

$$\omega \partial_y T = \begin{cases} \lambda_{\mathbb{I}} \partial_y h & \text{if } h \le h_{\mathbb{I}}^{\text{acl}} \\ \mathbf{0} & \text{if } h_{\mathbb{I}}^{\text{acl}} < h < h_{\mathfrak{g}}^{\text{acl}} \\ \lambda_{\mathfrak{g}} \partial_y h & \text{if } h \ge h_{\mathfrak{g}}^{\text{acl}} \end{cases}$$

• Since $\varrho_{\varepsilon}(h, \varphi^{\text{sol}}(h)) = \varrho(h)$ and $T_{\varepsilon}(h, \varphi^{\text{sol}}(h)) = T^{\text{sol}}$, formally 4-LMNC \longrightarrow (3)-LMNC

• In the 4-LMNC model the phase is not identified by the value of $h_{arepsilon}$ but $arphi_arepsilon$: we can have a mixture even if $h_arepsilon > h_{ ext{a}}^{ ext{ad}}$

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③ Relaxation Approach

$\text{4-}\mathrm{Lmnc}$

$$\begin{cases} \partial_{t} \varrho_{\varepsilon} + \partial_{y} (\varrho_{\varepsilon} v_{\varepsilon}) = 0\\ \partial_{t} (\varrho_{\varepsilon} h_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} h_{\varepsilon} v_{\varepsilon}) = \Phi + \partial_{y} (\omega_{\varepsilon} \partial_{y} T_{\varepsilon})\\ \partial_{t} (\varrho_{\varepsilon} \varphi_{\varepsilon}) + \partial_{y} (\varrho_{\varepsilon} \varphi_{\varepsilon} v_{\varepsilon}) = \frac{1}{\varepsilon} \varrho_{\varepsilon} \left(\varphi^{\text{out}} (h_{\varepsilon}) - \varphi_{\varepsilon} \right) \end{cases}$$

- Unknowns: h_{ε} , v_{ε} , φ_{ε} (mass fraction)
- EoS iso-Tp: $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto \varrho_{\varepsilon}$ and $(h_{\varepsilon}, \varphi_{\varepsilon}) \mapsto T_{\varepsilon}$
- Diffusion:

$$\omega_{\varepsilon}\partial_{y}T_{\varepsilon} = \frac{\omega_{\varepsilon}}{c_{p}(\varphi_{\varepsilon})}\partial_{y}h_{\varepsilon} + \omega_{\varepsilon}\frac{\partial T_{\varepsilon}}{\partial\varphi}\partial_{y}\varphi_{\varepsilon}$$



(3)-LMNC

$$\begin{cases} \partial_t \varrho + \partial_y(\varrho v) = 0\\ \partial_t(\varrho h) + \partial_y(\varrho h v) = \Phi + \partial_y(\omega \partial_y T) \end{cases}$$

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- \bullet EoS iso- $Tp\mu \!\!: h \mapsto \varrho$ and $h \mapsto T \rightsquigarrow$ mass fraction

$$arphi^{ ext{solt}}(h) \stackrel{ ext{def}}{=} egin{cases} 0 & ext{if } h \leq h_{\ell}^{ ext{solt}} \ rac{h - h_{\ell}^{ ext{solt}}}{h_{\ell}^{ ext{solt}} - h_{\ell}^{ ext{solt}}} & ext{if } h_{\ell}^{ ext{solt}} < h < h_{g}^{ ext{solt}} \ 1 & ext{if } h \geq h_{g}^{ ext{solt}} \end{cases}$$

Diffusion:

$$\omega \partial_y T = \begin{cases} \lambda_{\mathbb{I}} \partial_y h & \text{if } h \le h_{\mathbb{I}}^{\text{acl}} \\ \mathbf{0} & \text{if } h_{\mathbb{I}}^{\text{acl}} < h < h_{\mathfrak{g}}^{\text{acl}} \\ \lambda_{\mathfrak{g}} \partial_y h & \text{if } h \ge h_{\mathfrak{g}}^{\text{acl}} \end{cases}$$

• Since $\varrho_{\varepsilon}(h, \varphi^{\text{sat}}(h)) = \varrho(h)$ and $T_{\varepsilon}(h, \varphi^{\text{sat}}(h)) = T^{\text{sat}}$, formally 4-LMNC $\xrightarrow[\varepsilon \to 0]{}$ (3)-LMNC

• In the 4-LMNC model the phase is not identified by the value of h_{ε} but φ_{ε} : we can have a mixture even if $h_{\varepsilon} > h_{s}^{act}$.

Degener

3. Conclusion

3 Relaxation Approach – $\varepsilon = 10^{-12}$

③ Relaxation Approach – Validation?

- Asymptotic behavior?
- Transient behavior?
 - Plausible? 🗸
 - Comparison with an exact solution (in a simplified case) ?

 \rightsquigarrow focus on the exact transient solution in the form of a traveling wave as in Predictor-Corrector Approach

③ Relaxation Approach – Validation?

- Asymptotic behavior?
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 \rightsquigarrow focus on the exact transient solution in the form of a traveling wave as in Predictor-Corrector Approach

4. Full Model

- 4.1 The "real" Equation of state 4.2 The Full Model with Degenerating Diffusio
- 4.3 Concluding Remarks on the Full Model

Concluding Remarks on the Full Model

• Without diffusion or with non-degenerate diffusion: approaches as for Navier-Stokes equations

- Objective and the second se
 - Jumps in h, g, and v necessitate schemes based on conservative formulation...
 - ...but the system is not a classic hyperbolic system of conservation laws (no explicit time-derivative v)
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EoS

2. Degenerate

3. Conclusion

Concluding Remarks on the Full Model

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5. Summary



- Diffusive interface framework: two pure phases + mixture
- Hyperbolic conservation laws: HEM (mixture at saturation with phase transition)
- Low Mach Hypothesis: $p(t,{\bf x})=p_*+M^2\bar{p}(t,{\bf x})$ with $M\ll 1$

Steady-state model

- Mixture zone always exists without thermal diffusion
- Thermal diffusion in gas phase reduces the mixture zone, potentially causing it to vanish
- Connection to the Stefan problem (sharp interface)

Simplified model (constant density)

- Robust numerical schemes for enthalpy jump
- Precise asymptotic and plausible transient behavior observed in simulations
- Validation through traveling wave solution

Full model

- Numerical approaches similar to NS with or without non-degenerate thermal diffusion
- Challenges with degenerate diffusion, requiring conservative formulation despite it not being a hyperbolic system

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Appendix

6. Dimensionless Transformation

7. How to compute the traveling wave

6. Dimensionless Transformation



Dimensionless Transformation

- Define fixed reference values h^r and ϱ^r . We choice $h^r = q_m$ and $\varrho^r = \zeta_m / h^r$.
- \bullet Introduce dimensionless variables: $\bar{h}=\frac{h}{h^{r}}$ and $\bar{\Phi}=\frac{\Phi}{h^{r}\varrho^{r}}$
- Define dimensionless functions: $\bar{\varrho}(\bar{h}) = \frac{1}{\varrho^r} \varrho(h(\bar{h})) = \frac{1}{\varrho^r} \varrho(h^r \bar{h})$ and $\bar{\lambda}(\bar{h}) = \frac{1}{\varrho^r} \lambda(h(\bar{h})) = \frac{1}{\varrho^r} \lambda(h^r \bar{h})$

This yields transformed equations:

• $\partial_t(\varrho(h)) + \partial_y(\varrho(h)v) = 0 \rightsquigarrow \partial_t(\varrho(h^r\bar{h})) + \partial_y(\varrho(h^r\bar{h})v) = 0 \rightsquigarrow$

 $\partial_t \bar{\varrho} + \partial_y (\bar{\varrho} v) = 0$

•
$$\partial_t(\varrho(h)h) + \partial_y(\varrho(h)hv) = [\Phi + \partial_y(\lambda(h)\partial_yh)] \rightsquigarrow$$

 $\partial_t(h^r\bar{h}\varrho(h^r\bar{h})) + \partial_y(h^r\bar{h}\varrho(h^r\bar{h})v) = [\Phi + \partial_y(\lambda(h^r\bar{h})\partial_y(h^r\bar{h}))] \rightsquigarrow$
 $\partial_t(\bar{\varrho}\bar{h}) + \partial_y(\bar{\varrho}\bar{h}v) = \bar{\Phi} + \partial_y(\bar{\lambda}(\bar{h})\partial_y\bar{h})$

7. How to compute the traveling wave



How to compute the traveling wave – I

Consider only mixture/gas transition ($h_e(t) > h_{l}^{\text{sat}}$)

The EoS is

$$au(h) = rac{1}{arrho}(h) = egin{cases} rac{h-q_{arrho}}{\zeta_{arrho}} & ext{if } h > h_{arrho}^{ ext{salt}} \ rac{h-q_{arrho}}{\zeta_{armma}} & ext{otherwise} \ \end{cases}$$

• Constants given:
$$\lambda_{m}=0$$
, $\lambda_{g}>0$, q_{m} , q_{g} , ζ_{m} , ζ_{g} , $h_{g}^{
m sat}$

- Constants chosen:
 - c as the speed of the traveling wave
 - $y_q^{\rm sol}(0)$ as the initial jump position
 - $h_{
 m m}^{
 m salt} < h_{
 m g}^{
 m salt}$ as the bottom of jump
- \bullet Jump position displacement imposed: $y_{g}^{\rm sat}(t)=y_{g}^{\rm sat}(0)-ct$
- ullet Constant jump amplitude: $[\![h]\!]=h_g^{\rm salt}-h_m^{\rm salt}$ for all t

How to compute the traveling wave – II

We seek a solution in the form

Enthalpy

$$h(t,y) = \begin{cases} p_{\theta}(y - y_0(t)) + h_{\theta}^{\text{sat}} & \text{if } y > y_0(t) \\ p_{m}(y - y_0(t)) + h_{m}^{\text{sat}} & \text{otherwise} \end{cases} \quad \rightsquigarrow \quad h_e(t) = -p_m y_0(t) + h_m^{\text{sat}}.$$

Velocity (as a function of the enthalpy!)

$$v(h) = \begin{cases} a_{\theta}(h - h_{\theta}^{\text{soft}}) + v_{\theta}^{\text{soft}} & \text{if } h > h_s^{\text{soft}} \\ a_{\text{rm}}(h - h_{\text{rm}}^{\text{soft}}) + v_{\text{rm}}^{\text{soft}} & \text{otherwise} \end{cases} \quad \rightsquigarrow \quad v_e(t) = v(h_e(t)).$$

We denoted

- $\star_{\hat{g}}^{\text{solt}} = \star (h_{\hat{g}}^{\text{solt},+})$ as the value of \star at the top of jumps • $\star_{m}^{\text{solt}} = \star (h_{\hat{g}}^{\text{solt},-}) = \star_{\hat{g}}^{\text{solt}} - [\![\star]\!]$ as the value of \star at the bottom of jumps
- Initialization: we define $h_0(y) = h(t = 0, y)$ and $v_0(y) = v(h_0(y))$
- Constants to be determined: p_k , a_k , v_k^{sat}

How to compute the traveling wave - In Each Region

In each region, the functions are regular, and we can expand the partial derivatives:

$$\begin{cases} \partial_t \varrho + \partial_y (\varrho v) = 0\\ \partial_t (\varrho h) + \partial_y (\varrho h v) = [\Phi + \partial_y (\lambda_k \partial_y h)] \end{cases} \iff \begin{cases} \partial_t \tau + v \partial_y \tau = \tau \partial_y v\\ \partial_t h + v \partial_y h = \tau \left[\Phi + \partial_y (\lambda_k \partial_y h)\right] \end{cases}$$
$$\iff \begin{cases} \tau'(h) \left[\Phi + \partial_y (A_k \partial_y h)\right] = \partial_y v\\ \partial_t h + v \partial_y h = \tau \left[\Phi + \partial_y (\lambda_k \partial_y h)\right] \end{cases}$$

In each phase, h is affine, so the diffusion term is zero:

$$\begin{cases} \partial_y v = \Phi \tau'(h) \\ \partial_t h + v \partial_y h = \Phi \tau(h) \end{cases}$$

How to compute the traveling wave – Traveling Wave

We seek a solution of the form $h(t, y) = h_0(y - ct)$ and $v(t, y) = v_0(y - ct)$. The first equation $\partial_y v = \Phi \tau'(h)$ gives

$$a_k p_k = \Phi \frac{1}{\zeta_k}$$

As $\partial_t h = -c \partial_y h$, the second equation gives

$$(a_k(h - h_k^{\text{sol}}) + v_k^{\text{sol}} - c)p_k = \Phi \frac{h - q_k}{\zeta_k}$$

We express the velocity parameters a_k and $v_k^{\scriptscriptstyle{\rm Aal}}$ in terms of the enthalpy parameters

$$\begin{cases} a_k = \frac{\Phi}{p_k \zeta_k} \\ v_k^{\text{sat}} = c + \frac{\Phi}{p_k} \frac{(h_k^{\text{sat}} - q_k)}{\zeta_k} = c + \Phi \frac{\tau_k^{\text{sat}}}{p_k} \end{cases}$$

and we still need to fix the p_k .

How to compute the traveling wave – Jump relations

$$\begin{cases} -c\llbracket\varrho\rrbracket + \llbracket\varrho v\rrbracket = 0\\ -c\llbracket\varrho h\rrbracket + \llbracket\varrho vh\rrbracket = p_{g}A_{g}\end{cases}$$

The first jump relation becomes

$$(v_{g}^{\mathrm{sat}}-c)arrho_{g}^{\mathrm{sat}}=(v_{\mathrm{m}}^{\mathrm{sat}}-c)arrho_{\mathrm{m}}^{\mathrm{sat}}$$

and using the expression $v_k^{{}_{\rm Adt}}=c+\frac{\Phi}{p_k \varrho_k^{{}_{\rm Adt}}}$,

$$\frac{1}{p_{\vartheta}} = \frac{1}{p_{\scriptscriptstyle \rm TM}} \quad \Rightarrow \quad p_{\scriptscriptstyle \rm TM} = p_{\vartheta}$$

The second relation gives

$$(v_{\mathfrak{g}}^{\mathrm{sat}}-c)arrho_{\mathfrak{g}}^{\mathrm{sat}}h_{\mathfrak{g}}^{\mathrm{sat}}-(v_{m}^{\mathrm{sat}}-c)arrho_{m}^{\mathrm{sat}}h_{m}^{\mathrm{sat}}=p_{\mathfrak{g}}A_{\mathfrak{g}}$$

and using the expression $v_k^{\text{\tiny ant}}=c+\frac{\Phi}{p_k\varrho_k^{\text{\tiny ant}}}$ and the equality of slopes $p_{\text{\tiny TM}}=p_{\text{g}},$

$$p_{ heta} = p_{ extsf{m}} = \sqrt{rac{(h_{ heta}^{ extsf{ad}} - h_{ extsf{m}}^{ extsf{ad}})\Phi}{A_{ heta}}}$$
How to compute the traveling wave - Conclusion

Let's define $\eta \stackrel{\text{def}}{=} \sqrt{\frac{h_{\vartheta}^{\text{sol}} - h_{m}^{\text{sol}}}{\lambda_{\vartheta}/\Phi}}$. If $h_e(t) = h_m^{\text{sol}} - \eta y_{\vartheta}^{\text{sol}}(t)$ and $v_e(t) = v(h_e(t))$, the solution is given by

$$h(t,y) = \begin{cases} h_{\vartheta}^{\text{adt}} + \eta(y - y_{\vartheta}^{\text{adt}}(t)) & \text{if } y < y_{\vartheta}^{\text{adt}}(t) \\ h_{\vartheta}^{\text{adt}} + \eta(y - y_{\vartheta}^{\text{adt}}(t)) & \text{if } y > y_{\vartheta}^{\text{adt}}(t) \end{cases}$$

$$v(h) = \begin{cases} c + \frac{\Phi}{\eta \varrho(h_{m}^{s})} + \frac{\Phi}{\eta \zeta_{m}}(h - h_{m}^{\text{sol}}(t)) & \text{if } h < h_{m}^{\text{sol}}(t) \\ c + \frac{\Phi}{\eta \varrho(h_{g}^{s})} + \frac{\Phi}{\eta \zeta_{g}}(h - h_{g}^{\text{sol}}(t)) & \text{if } h > h_{g}^{\text{sol}}(t) \end{cases}$$