

Trento — April 20, 2009

Thesis Defence of **Gloria FACCANONI**

MODELLING AND SIMULATION OF LIQUID-VAPOR PHASE TRANSITION

A CONTRIBUTION TO THE STUDY OF BOILING CRISIS

Advisor at the École Polytechnique: G. Allaire
Advisor at the CEA-Saclay: S. Kokh
Advisor at the University of Trento: E. Toro



Outline

- 1 **Context**
- 2 **Model**
- 3 **Numerical Approximation**
- 4 **Numerical Examples**
- 5 **Conclusion**

Outline

1 Context

2 Model

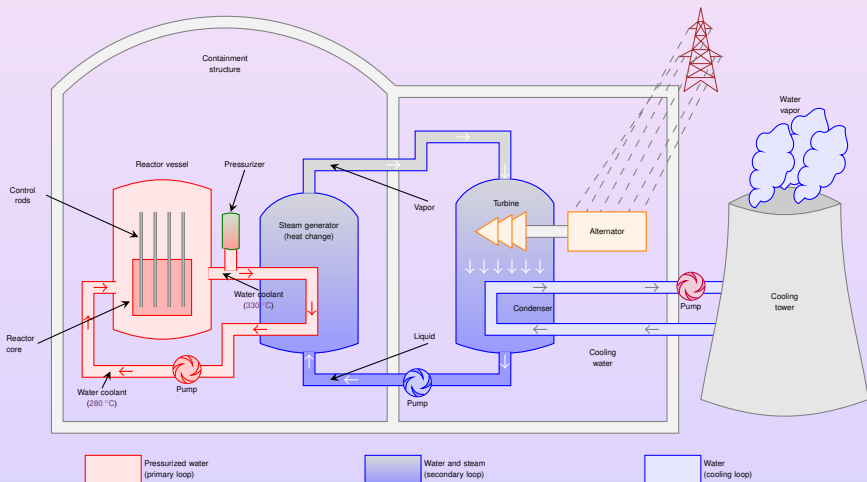
- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

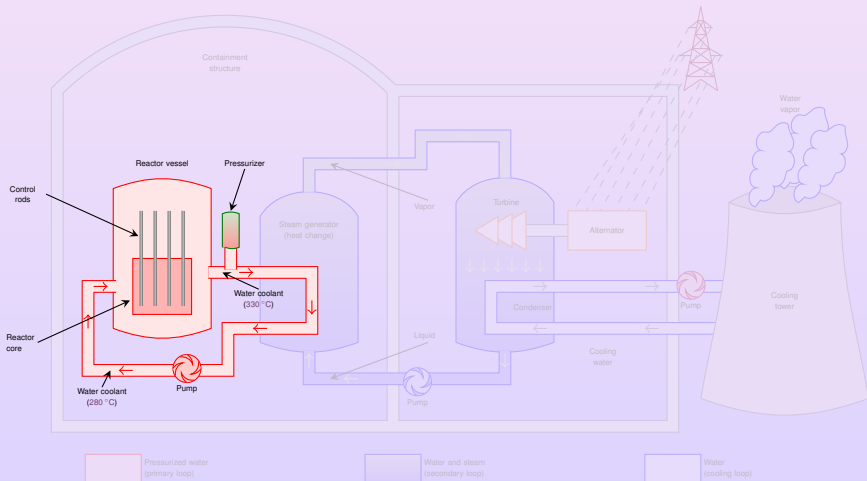
4 Numerical Examples

5 Conclusion

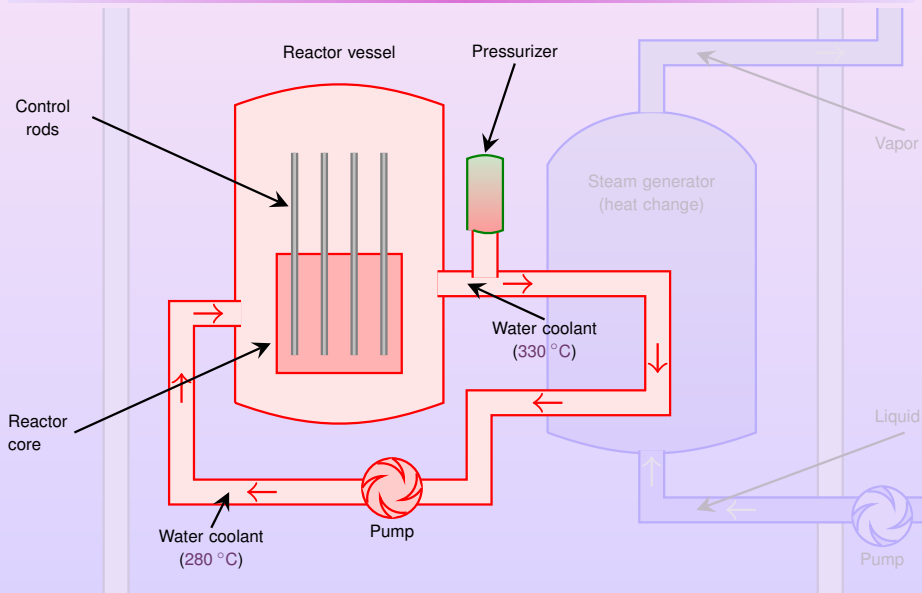
Pressurized Water Reactor



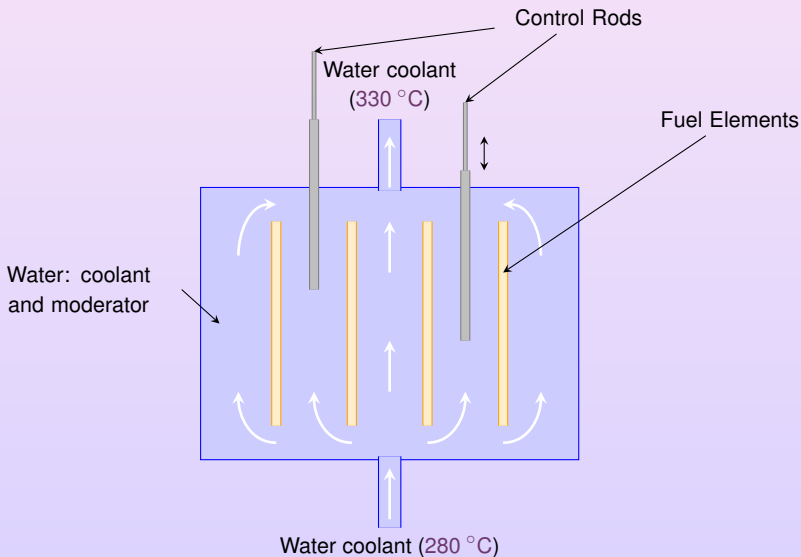
Pressurized Water Reactor



Pressurized Water Reactor



Core of a Pressurized Water Reactor

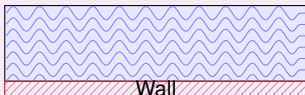


Boiling Crisis

PHENOMENON

Liquid phase heated by a wall at a fixed temperature T^{wall} .

When T^{wall} increases, we switch from a **Nucleate Boiling** to a **Film Boiling**.

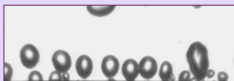
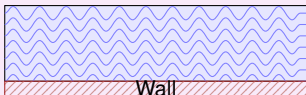


Boiling Crisis

PHENOMENON

Liquid phase heated by a wall at a fixed temperature T^{wall} .

When T^{wall} increases, we switch from a **Nucleate Boiling** to a **Film Boiling**.



Nucleate Boiling

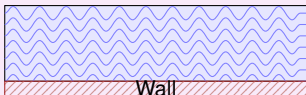
source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm

Boiling Crisis

PHENOMENON

Liquid phase heated by a wall at a fixed temperature T^{wall} .

When T^{wall} increases, we switch from a **Nucleate Boiling** to a **Film Boiling**.



Nucleate Boiling



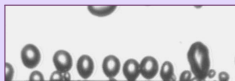
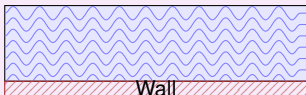
source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm

Boiling Crisis

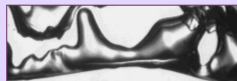
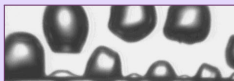
PHENOMENON

Liquid phase heated by a wall at a fixed temperature T^{wall} .

When T^{wall} increases, we switch from a **Nucleate Boiling** to a **Film Boiling**.



Nucleate Boiling



Film Boiling

source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm

OMEGA - CEA Grenoble



OMEGA - CEA Grenoble



Outline

1 Context

2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

“Ingredients” of the Model

✓ Simulating all bubbles,

- System of PDEs for the fluid flow (monophasic or diphasic),
- Phase transition (pressure and/or temperature variations),
- Heat Diffusion,
- Surface Tension,
- Gravity.

“Ingredients” of the Model

- ✓ Simulating all bubbles,
- ✓ System of PDEs for the fluid flow (monophasic or diphasic),
 - Phase transition (pressure and/or temperature variations),
 - Heat Diffusion,
 - Surface Tension,
 - Gravity.

“Ingredients” of the Model

- ✓ Simulating all bubbles,
- ✓ System of PDEs for the fluid flow (monophasic or diphasic),
- ✓ Phase transition (pressure and/or temperature variations),
 - Heat Diffusion,
 - Surface Tension,
 - Gravity.

“Ingredients” of the Model

- ✓ Simulating all bubbles,
- ✓ System of PDEs for the fluid flow (monophasic or diphasic),
- ✓ Phase transition (pressure and/or temperature variations),
- ✓ **Heat Diffusion,**
 - Surface Tension,
 - Gravity.

“Ingredients” of the Model

- ✓ Simulating all bubbles,
- ✓ System of PDEs for the fluid flow (monophasic or diphasic),
- ✓ Phase transition (pressure and/or temperature variations),
- ✓ Heat Diffusion,
- ✓ **Surface Tension,**
- Gravity.

“Ingredients” of the Model

- ✓ Simulating all bubbles,
- ✓ System of PDEs for the fluid flow (monophasic or diphasic),
- ✓ Phase transition (pressure and/or temperature variations),
- ✓ Heat Diffusion,
- ✓ Surface Tension,
- ✓ Gravity.

“Ingredients” of the Model

- ✓ Simulating all bubbles,
- ✓ System of PDEs for the fluid flow (monophasic or diphasic),
- ✓ Phase transition (pressure and/or temperature variations),
- ✓ Heat Diffusion,
- ✓ Surface Tension,
- ✓ Gravity.

Euler System

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}) = \mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}, \\ \partial_t\left(\rho\left(\frac{|\mathbf{u}|^2}{2} + \varepsilon\right)\right) + \operatorname{div}\left(\rho\left(\frac{|\mathbf{u}|^2}{2} + \varepsilon\right)\mathbf{u} + P \mathbf{u}\right) = (\mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}) \cdot \mathbf{u} - \operatorname{div}(q). \end{cases}$$

- $(\mathbf{x}, t) \mapsto \rho$ specific density,
- $(\mathbf{x}, t) \mapsto \varepsilon$ specific internal energy,
- $(\mathbf{x}, t) \mapsto \mathbf{u}$ velocity;
- $(\rho, \varepsilon) \mapsto \mathfrak{V}_{\text{vf}}$ body forces,
- $(\rho, \varepsilon) \mapsto \mathfrak{S}_{\text{sf}}$ surface forces,
- $(\rho, \varepsilon) \mapsto \operatorname{div}(q)$ heat transfer.

$(\rho, \varepsilon) \mapsto P$ pressure law.

Euler System

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + \mathbf{P} \mathbb{I}) = \mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}, \\ \partial_t\left(\rho\left(\frac{|\mathbf{u}|^2}{2} + \varepsilon\right)\right) + \operatorname{div}\left(\rho\left(\frac{|\mathbf{u}|^2}{2} + \varepsilon\right)\mathbf{u} + \mathbf{P} \mathbf{u}\right) = (\mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}) \cdot \mathbf{u} - \operatorname{div}(q). \end{cases}$$

- $(\mathbf{x}, t) \mapsto \rho$ specific density,
- $(\mathbf{x}, t) \mapsto \varepsilon$ specific internal energy,
- $(\mathbf{x}, t) \mapsto \mathbf{u}$ velocity;
- $(\rho, \varepsilon) \mapsto \mathfrak{V}_{\text{vf}}$ body forces,
- $(\rho, \varepsilon) \mapsto \mathfrak{S}_{\text{sf}}$ surface forces,
- $(\rho, \varepsilon) \mapsto \operatorname{div}(q)$ heat transfer.

$(\rho, \varepsilon) \mapsto P$ pressure law.

Outline

1 Context

2 Model

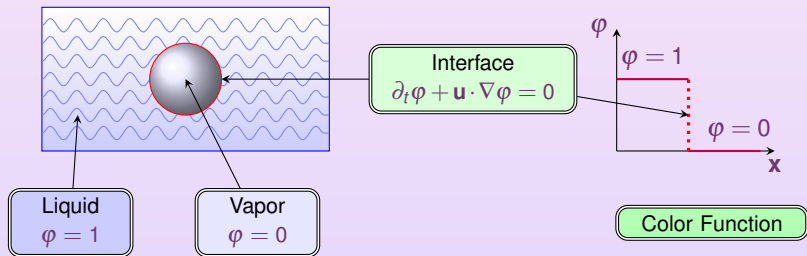
- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

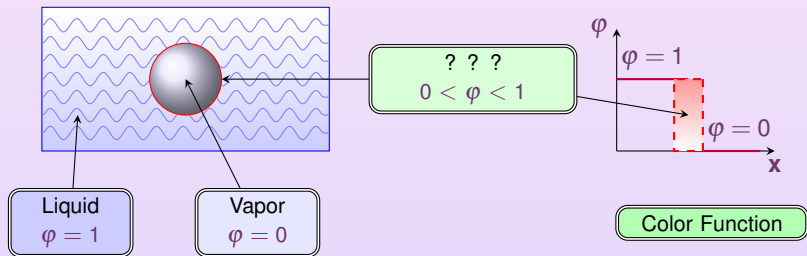
5 Conclusion

Liquid-Vapor Interface



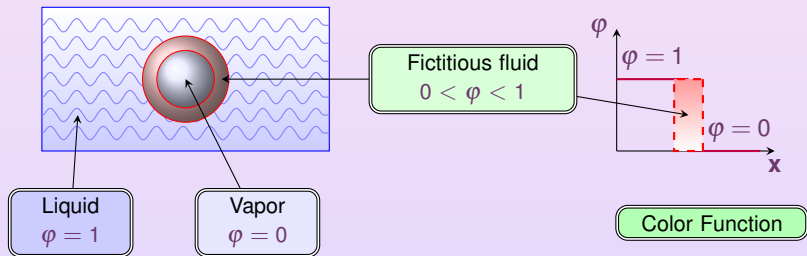
$$(\rho, \varepsilon) \mapsto P = \begin{cases} P^{\text{liq}} & \text{if } \varphi = 1; \\ P^{\text{vap}} & \text{if } \varphi = 0. \end{cases}$$

Liquid-Vapor Interface



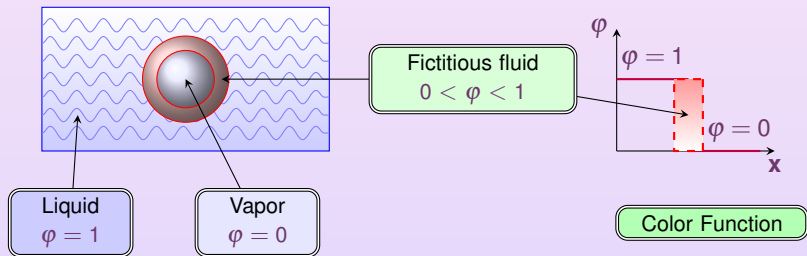
$$(\rho, \varepsilon) \mapsto P = \begin{cases} P^{\text{liq}} & \text{if } \varphi = 1; \\ ??? & \text{if } 0 < \varphi < 1; \\ P^{\text{vap}} & \text{if } \varphi = 0. \end{cases}$$

Liquid-Vapor Interface



$$(\rho, \varepsilon) \mapsto P = \begin{cases} P^{\text{liq}} & \text{if } \varphi = 1; \\ P^{\text{ff}} & \text{if } 0 < \varphi < 1; \\ P^{\text{vap}} & \text{if } \varphi = 0. \end{cases}$$

Liquid-Vapor Interface



➡ Goal: define a global pressure law such that

- $(\rho, \varepsilon, \mathbf{u}, P)$ are continuous (3 zones)
- the interface position and the phase change are implicit (i.e. $\nabla \varphi$)
- coherence with classical thermodynamics [H. CALLEN]

EOS of each PHASE $\alpha = 1, 2$

$$\left. \begin{array}{l} \tau_\alpha \text{ specific volume} \\ \varepsilon_\alpha \text{ specific internal energy} \end{array} \right\} \Rightarrow \mathbf{w}_\alpha \stackrel{\text{def}}{=} (\tau_\alpha, \varepsilon_\alpha);$$

$\mathbf{w}_\alpha \mapsto s_\alpha$ specific entropy (Hessian matrix neg. def.);

$$\begin{array}{l} \left\{ \begin{array}{l} T_\alpha \stackrel{\text{def}}{=} \left(\frac{\partial s_\alpha}{\partial \varepsilon_\alpha} \Big|_{\tau_\alpha} \right)^{-1} > 0 \quad \text{temperature,} \\ P_\alpha \stackrel{\text{def}}{=} T_\alpha \frac{\partial s_\alpha}{\partial \tau_\alpha} \Big|_{\varepsilon_\alpha} > 0 \quad \text{pressure,} \\ g_\alpha \stackrel{\text{def}}{=} \varepsilon_\alpha + P_\alpha \tau_\alpha - T_\alpha s_\alpha \quad \text{free enthalpy (Gibbs potential).} \end{array} \right. \end{array}$$

EOS of each PHASE $\alpha = 1, 2$

$$\left. \begin{array}{l} \tau_\alpha \text{ specific volume} \\ \varepsilon_\alpha \text{ specific internal energy} \end{array} \right\} \Rightarrow \mathbf{w}_\alpha \stackrel{\text{def}}{=} (\tau_\alpha, \varepsilon_\alpha);$$

$\mathbf{w}_\alpha \mapsto s_\alpha$ specific entropy (Hessian matrix neg. def.);

$$\begin{array}{l} \left\{ \begin{array}{l} T_\alpha \stackrel{\text{def}}{=} \left(\frac{\partial s_\alpha}{\partial \varepsilon_\alpha} \Big|_{\tau_\alpha} \right)^{-1} > 0 \quad \text{temperature,} \\ P_\alpha \stackrel{\text{def}}{=} T_\alpha \frac{\partial s_\alpha}{\partial \tau_\alpha} \Big|_{\varepsilon_\alpha} > 0 \quad \text{pressure,} \\ g_\alpha \stackrel{\text{def}}{=} \varepsilon_\alpha + P_\alpha \tau_\alpha - T_\alpha s_\alpha \quad \text{free enthalpy (Gibbs potential).} \end{array} \right. \end{array}$$

EOS of each PHASE $\alpha = 1, 2$

$$\left. \begin{array}{l} \tau_\alpha \text{ specific volume} \\ \varepsilon_\alpha \text{ specific internal energy} \end{array} \right\} \Rightarrow \mathbf{w}_\alpha \stackrel{\text{def}}{=} (\tau_\alpha, \varepsilon_\alpha);$$

$\mathbf{w}_\alpha \mapsto s_\alpha$ **specific entropy** (Hessian matrix neg. def.);

$$\left\{ \begin{array}{l} T_\alpha \stackrel{\text{def}}{=} \left(\frac{\partial s_\alpha}{\partial \varepsilon_\alpha} \Big|_{\tau_\alpha} \right)^{-1} > 0 \quad \text{temperature,} \\ P_\alpha \stackrel{\text{def}}{=} T_\alpha \frac{\partial s_\alpha}{\partial \tau_\alpha} \Big|_{\varepsilon_\alpha} > 0 \quad \text{pressure,} \\ g_\alpha \stackrel{\text{def}}{=} \varepsilon_\alpha + P_\alpha \tau_\alpha - T_\alpha s_\alpha \quad \text{free enthalpy (Gibbs potential).} \end{array} \right.$$

EOS of each PHASE $\alpha = 1, 2$

$$\left. \begin{array}{l} \tau_\alpha \text{ specific volume} \\ \varepsilon_\alpha \text{ specific internal energy} \end{array} \right\} \Rightarrow \mathbf{w}_\alpha \stackrel{\text{def}}{=} (\tau_\alpha, \varepsilon_\alpha);$$

$\mathbf{w}_\alpha \mapsto s_\alpha$ specific entropy (Hessian matrix neg. def.);

$$\left\{ \begin{array}{l} T_\alpha \stackrel{\text{def}}{=} \left(\frac{\partial s_\alpha}{\partial \varepsilon_\alpha} \Big|_{\tau_\alpha} \right)^{-1} > 0 \quad \text{temperature,} \\ P_\alpha \stackrel{\text{def}}{=} T_\alpha \frac{\partial s_\alpha}{\partial \tau_\alpha} \Big|_{\varepsilon_\alpha} > 0 \quad \text{pressure,} \\ g_\alpha \stackrel{\text{def}}{=} \varepsilon_\alpha + P_\alpha \tau_\alpha - T_\alpha s_\alpha \quad \text{free enthalpy (Gibbs potential).} \end{array} \right.$$

EOS of the Mixture

- $\mathbf{w} \stackrel{\text{def}}{=} y\mathbf{w}_1 + (1 - y)\mathbf{w}_2$;
- y mass fraction;
- z volume fraction s.t. $y\tau_1 = z\tau$;
- ψ energy fraction s.t. $y\varepsilon_1 = \psi\varepsilon$.

EOS of the Mixture

- $\mathbf{w} \stackrel{\text{def}}{=} y\mathbf{w}_1 + (1-y)\mathbf{w}_2$;
- y mass fraction;
- z volume fraction s.t. $y\tau_1 = z\tau$;
- ψ energy fraction s.t. $y\varepsilon_1 = \psi\varepsilon$.

ENTROPY WITHOUT PHASE CHANGE

$$\sigma \stackrel{\text{def}}{=} ys_1(\mathbf{w}_1) + (1-y)s_2(\mathbf{w}_2) = ys_1\left(\frac{z}{y}\tau, \frac{\psi}{y}\varepsilon\right) + (1-y)s_2\left(\frac{1-z}{1-y}\tau, \frac{1-\psi}{1-y}\varepsilon\right)$$

$$P = \left(\frac{\partial \sigma}{\partial \varepsilon} \Big|_{\tau; y, z, \psi} \right)^{-1} \frac{\partial \sigma}{\partial \tau} \Big|_{\varepsilon; y, z, \psi}$$

EOS of the Mixture

- $\mathbf{w} \stackrel{\text{def}}{=} y\mathbf{w}_1 + (1-y)\mathbf{w}_2$;
- y mass fraction;
- z volume fraction s.t. $y\tau_1 = z\tau$;
- ψ energy fraction s.t. $y\varepsilon_1 = \psi\varepsilon$.

ENTROPY WITHOUT PHASE CHANGE

$$\sigma \stackrel{\text{def}}{=} ys_1(\mathbf{w}_1) + (1-y)s_2(\mathbf{w}_2) = ys_1\left(\frac{z}{y}\tau, \frac{\psi}{y}\varepsilon\right) + (1-y)s_2\left(\frac{1-z}{1-y}\tau, \frac{1-\psi}{1-y}\varepsilon\right)$$

$$P = \left(\frac{\partial \sigma}{\partial \varepsilon} \Big|_{\tau; y, z, \psi} \right)^{-1} \frac{\partial \sigma}{\partial \tau} \Big|_{\varepsilon; y, z, \psi}$$

EOS of the Mixture

- $\mathbf{w} \stackrel{\text{def}}{=} y\mathbf{w}_1 + (1 - y)\mathbf{w}_2$;
- y mass fraction;
- z volume fraction s.t. $y\tau_1 = z\tau$;
- ψ energy fraction s.t. $y\varepsilon_1 = \psi\varepsilon$.

ENTROPY WITHOUT PHASE CHANGE

$$\sigma \stackrel{\text{def}}{=} ys_1(\mathbf{w}_1) + (1 - y)s_2(\mathbf{w}_2) = ys_1\left(\frac{z}{y}\tau, \frac{\psi}{y}\varepsilon\right) + (1 - y)s_2\left(\frac{1-z}{1-y}\tau, \frac{1-\psi}{1-y}\varepsilon\right)$$

$$P = \left(\frac{\partial \sigma}{\partial \varepsilon} \Big|_{\tau; y, z, \psi} \right)^{-1} \frac{\partial \sigma}{\partial \tau} \Big|_{\varepsilon; y, z, \psi}$$

Outline

1 Context

2 Model

- Equation of State WITHOUT Phase Change
- **Equation of State WITH Phase Change**
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

EOS with Phase Change

ENTROPY WITHOUT PH.CH.

$$(\mathbf{w}, z, y, \psi) \mapsto \sigma$$



ENTROPY AT EQUILIBRIUM

$$\mathbf{w} \mapsto s^{\text{eq}}$$

DEFINITION [H. CALLEN, PH. HELLUY ...]

Optimization Problem:

$$s^{\text{eq}}(\mathbf{w}) \stackrel{\text{def}}{=} \max_{z, y, \psi \in [0, 1]^3} \sigma(\mathbf{w}, z, y, \psi)$$

Optimality Condition:
$$\begin{cases} T_1(z, y, \psi) = T_2(z, y, \psi) \\ P_1(z, y, \psi) = P_2(z, y, \psi) \\ g_1(z, y, \psi) = g_2(z, y, \psi) \\ z, y, \psi \in]0, 1[^3 \end{cases}$$

Solution: (z^*, y^*, ψ^*)

EOS with Phase Change

ENTROPY WITHOUT PH.CH.

$$(\mathbf{w}, z, y, \psi) \mapsto \sigma$$



ENTROPY AT EQUILIBRIUM

$$\mathbf{w} \mapsto s^{\text{eq}}$$

DEFINITION [H. CALLEN, PH. HELLUY ...]

Optimization Problem:

$$s^{\text{eq}}(\mathbf{w}) \stackrel{\text{def}}{=} \max_{z, y, \psi \in [0, 1]^3} \sigma(\mathbf{w}, z, y, \psi)$$

Optimality Condition:
$$\begin{cases} T_1(z, y, \psi) = T_2(z, y, \psi) \\ P_1(z, y, \psi) = P_2(z, y, \psi) \\ g_1(z, y, \psi) = g_2(z, y, \psi) \\ z, y, \psi \in]0, 1[^3 \end{cases}$$

Solution: (z^*, y^*, ψ^*)

EOS with Phase Change

ENTROPY WITHOUT PH.CH.

$$(\mathbf{w}, z, y, \psi) \mapsto \sigma$$



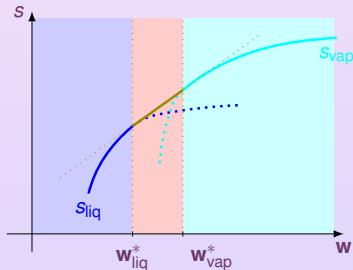
ENTROPY AT EQUILIBRIUM

$$\mathbf{w} \mapsto s^{\text{eq}}$$

DEFINITION [H. CALLEN, PH. HELLUY ...]

Optimization Problem:

$$\begin{aligned} s^{\text{eq}}(\mathbf{w}) &\stackrel{\text{def}}{=} \max_{z, y, \psi \in [0, 1]^3} \sigma(\mathbf{w}, z, y, \psi) \\ &= \text{co} \{ \max \{ s_1(\mathbf{w}), s_2(\mathbf{w}) \} \} \end{aligned}$$



Optimality Condition:

$$\begin{cases} T_1(z, y, \psi) = T_2(z, y, \psi) \\ P_1(z, y, \psi) = P_2(z, y, \psi) \\ g_1(z, y, \psi) = g_2(z, y, \psi) \\ z, y, \psi \in]0, 1[^3 \end{cases}$$

Solution: (z^*, y^*, ψ^*)

EOS with Phase Change

ENTROPY WITHOUT PH.CH.

$$(\mathbf{w}, z, y, \psi) \mapsto \sigma$$



ENTROPY AT EQUILIBRIUM

$$\mathbf{w} \mapsto s^{\text{eq}}$$

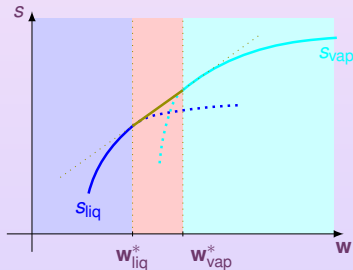
DEFINITION [H. CALLEN, PH. HELLUY ...]

Optimization Problem:

$$\begin{aligned} s^{\text{eq}}(\mathbf{w}) &\stackrel{\text{def}}{=} \max_{z, y, \psi \in [0, 1]^3} \sigma(\mathbf{w}, z, y, \psi) \\ &= \text{co} \{ \max \{ s_1(\mathbf{w}), s_2(\mathbf{w}) \} \} \end{aligned}$$

Optimality Condition:

$$\begin{cases} T_1(z, y, \psi) = T_2(z, y, \psi) \\ P_1(z, y, \psi) = P_2(z, y, \psi) \\ g_1(z, y, \psi) = g_2(z, y, \psi) \\ z, y, \psi \in]0, 1[^3 \end{cases}$$



Solution: (z^*, y^*, ψ^*)

EOS with Phase Change

ENTROPY WITHOUT PH.CH.

$$(\mathbf{w}, z, y, \psi) \mapsto \sigma$$



ENTROPY AT EQUILIBRIUM

$$\mathbf{w} \mapsto s^{\text{eq}}$$

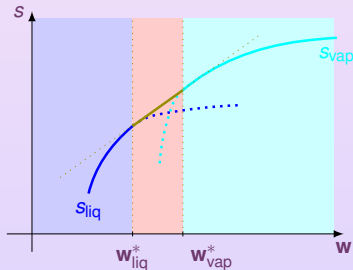
DEFINITION [H. CALLEN, PH. HELLUY ...]

Optimization Problem:

$$\begin{aligned} s^{\text{eq}}(\mathbf{w}) &\stackrel{\text{def}}{=} \max_{z, y, \psi \in [0, 1]^3} \sigma(\mathbf{w}, z, y, \psi) \\ &= \text{co} \{ \max \{ s_1(\mathbf{w}), s_2(\mathbf{w}) \} \} \end{aligned}$$

Optimality Condition:

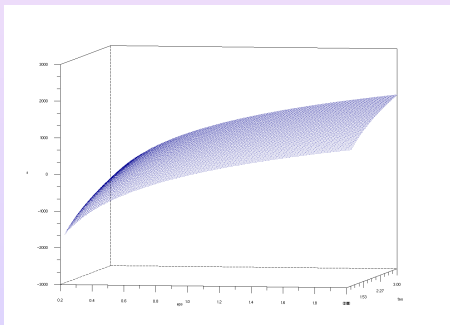
$$\begin{cases} T_1(z, y, \psi) = T_2(z, y, \psi) \\ P_1(z, y, \psi) = P_2(z, y, \psi) \\ g_1(z, y, \psi) = g_2(z, y, \psi) \\ z, y, \psi \in]0, 1[^3 \end{cases}$$



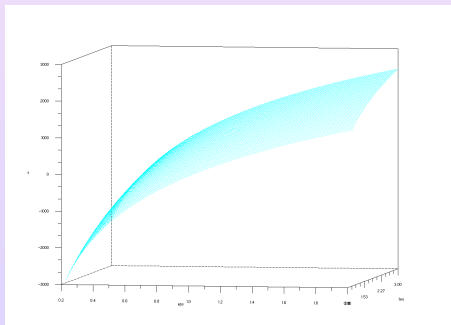
Solution: (z^*, y^*, ψ^*)

Concave Hull with two Perfect Gases

$$(\tau, \varepsilon) \mapsto s_{\text{liq}}$$

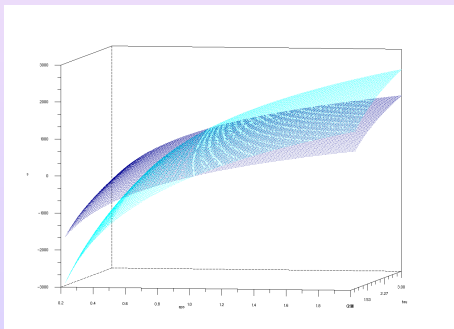


$$(\tau, \varepsilon) \mapsto s_{\text{vap}}$$



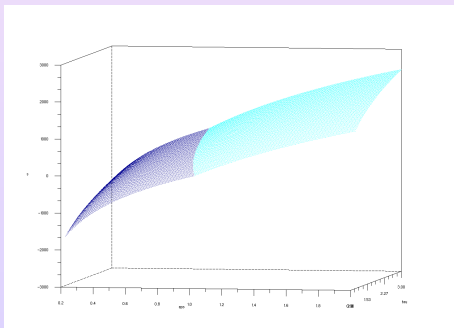
Concave Hull with two Perfect Gases

$$(\tau, \varepsilon) \mapsto \max\{s_{\text{liq}}, s_{\text{vap}}\}$$



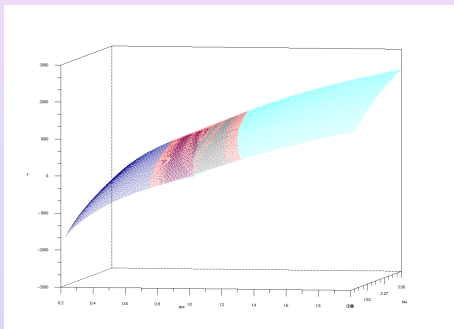
Concave Hull with two Perfect Gases

$$(\tau, \varepsilon) \mapsto \max\{s_{\text{liq}}, s_{\text{vap}}\}$$



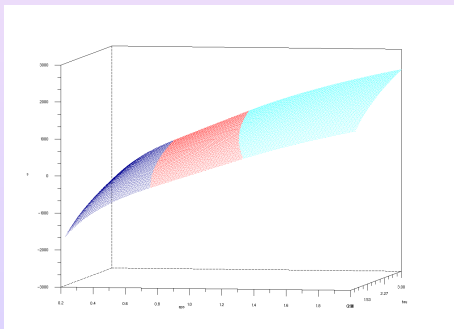
Concave Hull with two Perfect Gases

$$(\tau, \varepsilon) \mapsto \text{co} \left\{ \max \{ s_{\text{liq}}, s_{\text{vap}} \} \right\}$$



Concave Hull with two Perfect Gases

$$(\tau, \varepsilon) \mapsto \text{co} \left\{ \max \{ s_{\text{liq}}, s_{\text{vap}} \} \right\}$$



From $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ to $\mathbf{w} \mapsto P^{\text{eq}}$

For all $\tilde{\mathbf{w}}$ fixed, we seek $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$ as the solution of the system

$$\begin{cases} P_{\text{liq}}(\mathbf{w}_{\text{liq}}) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ T_{\text{liq}}(\mathbf{w}_{\text{liq}}) = T_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ g_{\text{liq}}(\mathbf{w}_{\text{liq}}) = g_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ \tilde{\mathbf{w}} = y\mathbf{w}_{\text{liq}} + (1-y)\mathbf{w}_{\text{vap}} \end{cases}$$

- if $y^* \in]0, 1[$ then $\tilde{\mathbf{w}}$ is an **equilibrium mixture state**

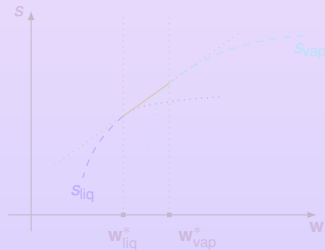
$$\mathbf{s}^{\text{eq}}(\tilde{\mathbf{w}}) = y^* \mathbf{s}_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) + (1-y^*) \mathbf{s}_{\text{vap}}(\mathbf{w}_{\text{vap}}^*),$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}^*);$$

- if the system has no solution or $y^* \notin]0, 1[$ then $\tilde{\mathbf{w}}$ is a **monophasic pure state**

$$\mathbf{s}^{\text{eq}}(\tilde{\mathbf{w}}) = \max\{\mathbf{s}_{\text{liq}}(\tilde{\mathbf{w}}), \mathbf{s}_{\text{vap}}(\tilde{\mathbf{w}})\},$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\tilde{\mathbf{w}})$$



From $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ to $\mathbf{w} \mapsto P^{\text{eq}}$

For all $\tilde{\mathbf{w}}$ fixed, we seek $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$ as the solution of the system

$$\begin{cases} P_{\text{liq}}(\mathbf{w}_{\text{liq}}) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ T_{\text{liq}}(\mathbf{w}_{\text{liq}}) = T_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ g_{\text{liq}}(\mathbf{w}_{\text{liq}}) = g_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ \tilde{\mathbf{w}} = y\mathbf{w}_{\text{liq}} + (1-y)\mathbf{w}_{\text{vap}} \end{cases}$$

- if $y^* \in]0, 1[$ then $\tilde{\mathbf{w}}$ is an **equilibrium mixture state**

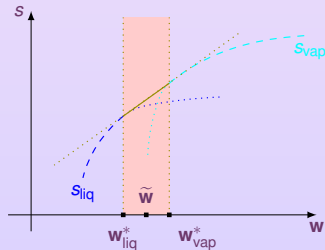
$$s^{\text{eq}}(\tilde{\mathbf{w}}) = y^* s_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) + (1-y^*) s_{\text{vap}}(\mathbf{w}_{\text{vap}}^*),$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}^*);$$

- if the system has no solution or $y^* \notin]0, 1[$ then $\tilde{\mathbf{w}}$ is a **monophasic pure state**

$$s^{\text{eq}}(\tilde{\mathbf{w}}) = \max\{s_{\text{liq}}(\tilde{\mathbf{w}}), s_{\text{vap}}(\tilde{\mathbf{w}})\},$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\tilde{\mathbf{w}})$$



From $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ to $\mathbf{w} \mapsto P^{\text{eq}}$

For all $\tilde{\mathbf{w}}$ fixed, we seek $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$ as the solution of the system

$$\begin{cases} P_{\text{liq}}(\mathbf{w}_{\text{liq}}) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ T_{\text{liq}}(\mathbf{w}_{\text{liq}}) = T_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ g_{\text{liq}}(\mathbf{w}_{\text{liq}}) = g_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ \tilde{\mathbf{w}} = y\mathbf{w}_{\text{liq}} + (1-y)\mathbf{w}_{\text{vap}} \end{cases}$$

- 1 if $y^* \in]0, 1[$ then $\tilde{\mathbf{w}}$ is an **equilibrium mixture state**

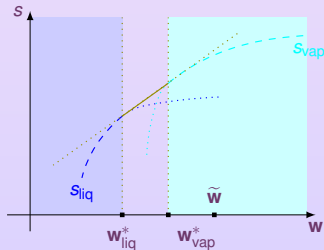
$$\mathbf{s}^{\text{eq}}(\tilde{\mathbf{w}}) = y^* \mathbf{s}_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) + (1-y^*) \mathbf{s}_{\text{vap}}(\mathbf{w}_{\text{vap}}^*),$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}^*);$$

- 2 if the system has no solution or $y^* \notin]0, 1[$ then $\tilde{\mathbf{w}}$ is a **monophasic pure state**

$$\mathbf{s}^{\text{eq}}(\tilde{\mathbf{w}}) = \max\{\mathbf{s}_{\text{liq}}(\tilde{\mathbf{w}}), \mathbf{s}_{\text{vap}}(\tilde{\mathbf{w}})\},$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\alpha}(\tilde{\mathbf{w}}).$$



From $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ to $\mathbf{w} \mapsto P^{\text{eq}}$

For all $\tilde{\mathbf{w}}$ fixed, we seek $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$ as the solution of the system

$$\begin{cases} P_{\text{liq}}(\mathbf{w}_{\text{liq}}) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ T_{\text{liq}}(\mathbf{w}_{\text{liq}}) = T_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ g_{\text{liq}}(\mathbf{w}_{\text{liq}}) = g_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ \tilde{\mathbf{w}} = y\mathbf{w}_{\text{liq}} + (1-y)\mathbf{w}_{\text{vap}} \end{cases}$$

- 1 if $y^* \in]0, 1[$ then $\tilde{\mathbf{w}}$ is an **equilibrium mixture state**

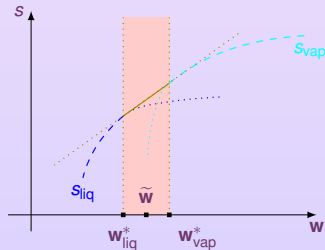
$$s^{\text{eq}}(\tilde{\mathbf{w}}) = y^* s_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) + (1-y^*) s_{\text{vap}}(\mathbf{w}_{\text{vap}}^*),$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}^*);$$

- 2 if the system has no solution or $y^* \notin]0, 1[$ then $\tilde{\mathbf{w}}$ is a **monophasic pure state**

$$s^{\text{eq}}(\tilde{\mathbf{w}}) = \max\{s_{\text{liq}}(\tilde{\mathbf{w}}), s_{\text{vap}}(\tilde{\mathbf{w}})\},$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\alpha}(\tilde{\mathbf{w}}).$$



From $\mathbf{w} \mapsto \mathbf{s}^{\text{eq}}$ to $\mathbf{w} \mapsto P^{\text{eq}}$

For all $\tilde{\mathbf{w}}$ fixed, we seek $(\mathbf{w}_{\text{liq}}^*, \mathbf{w}_{\text{vap}}^*, y^*)$ as the solution of the system

$$\begin{cases} P_{\text{liq}}(\mathbf{w}_{\text{liq}}) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ T_{\text{liq}}(\mathbf{w}_{\text{liq}}) = T_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ g_{\text{liq}}(\mathbf{w}_{\text{liq}}) = g_{\text{vap}}(\mathbf{w}_{\text{vap}}) \\ \tilde{\mathbf{w}} = y\mathbf{w}_{\text{liq}} + (1-y)\mathbf{w}_{\text{vap}} \end{cases}$$

- 1 if $y^* \in]0, 1[$ then $\tilde{\mathbf{w}}$ is an **equilibrium mixture state**

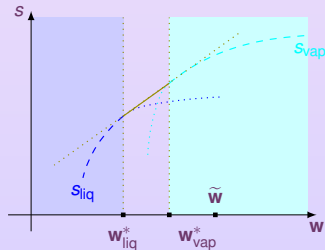
$$s^{\text{eq}}(\tilde{\mathbf{w}}) = y^* s_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) + (1-y^*) s_{\text{vap}}(\mathbf{w}_{\text{vap}}^*),$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\text{liq}}(\mathbf{w}_{\text{liq}}^*) = P_{\text{vap}}(\mathbf{w}_{\text{vap}}^*);$$

- 2 if the system has no solution or $y^* \notin]0, 1[$ then $\tilde{\mathbf{w}}$ is a **monophasic pure state**

$$s^{\text{eq}}(\tilde{\mathbf{w}}) = \max\{s_{\text{liq}}(\tilde{\mathbf{w}}), s_{\text{vap}}(\tilde{\mathbf{w}})\},$$

$$P^{\text{eq}}(\tilde{\mathbf{w}}) = P_{\alpha}(\tilde{\mathbf{w}}).$$



Summary of the Model

$$\mathbf{w} \mapsto S^{\text{eq}}$$

$$\begin{cases} g_1(\mathbf{w}_1) = g_2(\mathbf{w}_2) \\ P_1(\mathbf{w}_1) = P_2(\mathbf{w}_2) \\ T_1(\mathbf{w}_1) = T_2(\mathbf{w}_2) \\ \mathbf{w} = y\mathbf{w}_1 + (1-y)\mathbf{w}_2 \end{cases}$$

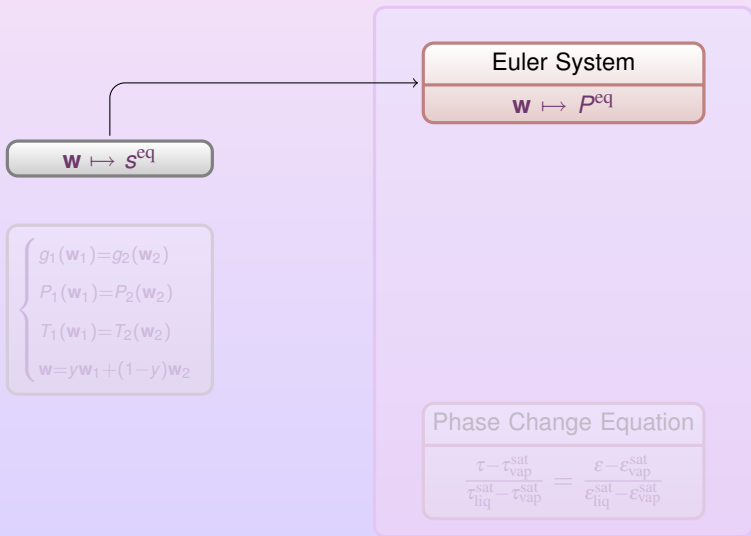
Euler System

$$\mathbf{w} \mapsto P^{\text{eq}}$$

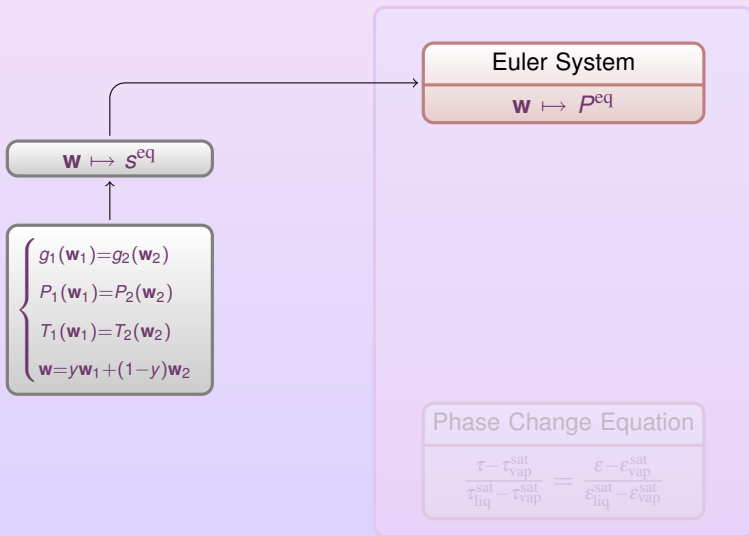
Phase Change Equation

$$\frac{\tau - \tau_{\text{vap}}^{\text{sat}}}{\tau_{\text{liq}}^{\text{sat}} - \tau_{\text{vap}}^{\text{sat}}} = \frac{\varepsilon - \varepsilon_{\text{vap}}^{\text{sat}}}{\varepsilon_{\text{liq}}^{\text{sat}} - \varepsilon_{\text{vap}}^{\text{sat}}}$$

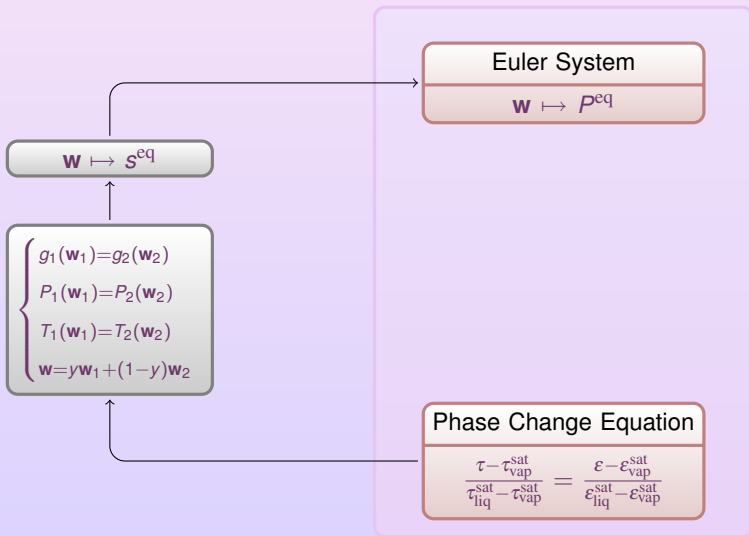
Summary of the Model



Summary of the Model



Summary of the Model



Summary of the Model

$$\mathbf{w} \mapsto S^{\text{eq}}$$

$$\begin{cases} g_1(\mathbf{w}_1) = g_2(\mathbf{w}_2) \\ P_1(\mathbf{w}_1) = P_2(\mathbf{w}_2) \\ T_1(\mathbf{w}_1) = T_2(\mathbf{w}_2) \\ \mathbf{w} = y\mathbf{w}_1 + (1-y)\mathbf{w}_2 \end{cases}$$

Euler System

$$\mathbf{w} \mapsto P^{\text{eq}}$$

Phase Change Equation

$$\frac{\tau - \tau_{\text{vap}}^{\text{sat}}}{\tau_{\text{liq}}^{\text{sat}} - \tau_{\text{vap}}^{\text{sat}}} = \frac{\varepsilon - \varepsilon_{\text{vap}}^{\text{sat}}}{\varepsilon_{\text{liq}}^{\text{sat}} - \varepsilon_{\text{vap}}^{\text{sat}}}$$

Outline

1 Context

2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- **The Phase Change Equation**
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

Analytical EOS

(τ, ε) fixed

$(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$ SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

(P, T) SOLUTION OF

$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \frac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \frac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

$$T \mapsto P = P^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(P^{\text{sat}}(T), T)$$

→ more examples

Analytical EOS

(τ, ε) fixed

$(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$ SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

(P, T) SOLUTION OF

$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \frac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \frac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

$$T \mapsto P = P^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(P^{\text{sat}}(T), T)$$

→ More Examples

Analytical EOS

(τ, ε) fixed

$(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$ SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

(P, T) SOLUTION OF

$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \frac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \frac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

$$T \mapsto P = P^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(P^{\text{sat}}(T), T)$$

→ next example

Analytical EOS

(τ, ε) fixed

$(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$ SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

(P, T) SOLUTION OF

$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \frac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \frac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

$$T \mapsto P = P^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(P^{\text{sat}}(T), T)$$

► [next example](#)

Analytical EOS

(τ, ε) fixed

$(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$ SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

(P, T) SOLUTION OF

$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \frac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \frac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

least square approximation

$$T \mapsto P = \hat{P}^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{where} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \stackrel{\text{def}}{=} \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(\hat{P}^{\text{sat}}(T), T)$$

► Water Example

Tabulated EOS

T (K)	P^{sat} (MPa)	Volume (m^3/kg)		Internal Energy (kJ/kg)	
		$\tau_{\text{liq}}^{\text{sat}}$	$\tau_{\text{vap}}^{\text{sat}}$	$\epsilon_{\text{liq}}^{\text{sat}}$	$\epsilon_{\text{vap}}^{\text{sat}}$
275	0,00069845	0,0010001	181,60	7,7590	2377,5
278	0,00086349	0,0010001	148,48	20,388	2381,6
281	0,0010621	0,0010002	122,01	32,996	2385,7
284	0,0012999	0,0010004	100,74	45,586	2389,8
287	0,0015835	0,0010008	83,560	58,162	2393,9
290	0,0019200	0,0010012	69,625	70,727	2398,0
293	0,0023177	0,0010018	58,267	83,284	2402,1
296	0,0027856	0,0010025	48,966	95,835	2406,2
299	0,0033342	0,0010032	41,318	108,38	2410,3
302	0,0039745	0,0010041	35,002	120,92	2414,4
305	0,0047193	0,0010050	29,764	133,46	2418,4
308	0,0055825	0,0010060	25,403	146	2422,5
...

Source: <http://webbook.nist.gov/chemistry/fluid/>

Tabulated EOS

(τ, ε) fixed

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{with} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \quad \text{tabulated}$$

\rightsquigarrow

$$\frac{\tau - \widehat{\tau}_2^{\text{sat}}(T)}{\widehat{\tau}_1^{\text{sat}}(T) - \widehat{\tau}_2^{\text{sat}}(T)} = \frac{\varepsilon - \widehat{\varepsilon}_2^{\text{sat}}(T)}{\widehat{\varepsilon}_1^{\text{sat}}(T) - \widehat{\varepsilon}_2^{\text{sat}}(T)} \quad \text{with} \quad \begin{pmatrix} \widehat{\tau} \\ \widehat{\varepsilon} \end{pmatrix}_{\alpha}^{\text{sat}}(T)$$

★ Water Examples

Tabulated EOS

(τ, ε) fixed

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)}$$

with $\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T)$ tabulated

$\}} \leftarrow$

$$\frac{\tau - \widehat{\tau}_2^{\text{sat}}(T)}{\widehat{\tau}_1^{\text{sat}}(T) - \widehat{\tau}_2^{\text{sat}}(T)} = \frac{\varepsilon - \widehat{\varepsilon}_2^{\text{sat}}(T)}{\widehat{\varepsilon}_1^{\text{sat}}(T) - \widehat{\varepsilon}_2^{\text{sat}}(T)}$$

with $\begin{pmatrix} \widehat{\tau} \\ \widehat{\varepsilon} \end{pmatrix}_{\alpha}^{\text{sat}}(T)$

least square approximations

★ Water Examples

Tabulated EOS

(τ, ε) fixed

T SOLUTION OF

$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)} \quad \text{with} \quad \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) \quad \text{tabulated}$$

$\}} \leftarrow$

$$\frac{\tau - \widehat{\tau}_2^{\text{sat}}(T)}{\widehat{\tau}_1^{\text{sat}}(T) - \widehat{\tau}_2^{\text{sat}}(T)} = \frac{\varepsilon - \widehat{\varepsilon}_2^{\text{sat}}(T)}{\widehat{\varepsilon}_1^{\text{sat}}(T) - \widehat{\varepsilon}_2^{\text{sat}}(T)} \quad \text{with} \quad \begin{pmatrix} \widehat{\tau} \\ \widehat{\varepsilon} \end{pmatrix}_{\alpha}^{\text{sat}}(T)$$

least square approximations

► Water Examples

Phase Change Equation: Summary

PHASE CHANGE EQUATION

$$\frac{\tau - \tau_{\text{vap}}^{\text{sat}}}{\tau_{\text{liq}}^{\text{sat}} - \tau_{\text{vap}}^{\text{sat}}} = \frac{\varepsilon - \varepsilon_{\text{vap}}^{\text{sat}}}{\varepsilon_{\text{liq}}^{\text{sat}} - \varepsilon_{\text{vap}}^{\text{sat}}}$$

with

$$T \mapsto \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(T) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(T, P^{\text{sat}}(T))$$

or

$$P \mapsto \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}^{\text{sat}}(P) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_{\alpha}(T^{\text{sat}}(P), P)$$

Phase Change Equation: Summary

How to compute saturation functions τ_α^{sat} and $\varepsilon_\alpha^{\text{sat}}$

- Analytical EOS:** we compute the saturation functions τ_α^{sat} and $\varepsilon_\alpha^{\text{sat}}$ by the **Coexistence Curve**:

- Exact: $T \mapsto P^{\text{sat}}(T)$ or $P \mapsto T^{\text{sat}}(P)$

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(P) = \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha(T^{\text{sat}}(P), P) \quad \text{e.g. Simplified Stiffened Gases}$$

- Approximated: $T \mapsto \hat{P}^{\text{sat}}(T) \approx P^{\text{sat}}(T)$

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(T) \approx \begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha(T, \hat{P}^{\text{sat}}(T)) \quad \text{e.g. General Stiffened Gases}$$

- Tabulated EOS:** the saturation functions τ_α^{sat} and $\varepsilon_\alpha^{\text{sat}}$ **are given** by experiments and we set

$$\begin{pmatrix} \tau \\ \varepsilon \end{pmatrix}_\alpha^{\text{sat}}(T \text{ or } P) \approx \begin{pmatrix} \hat{\tau} \\ \hat{\varepsilon} \end{pmatrix}_\alpha^{\text{sat}}(T \text{ or } P)$$

Outline

1 Context

2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- **Conservation Laws**

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

Dynamic Liquid-Vapor Phase Change

EULER SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \right) + \operatorname{div} \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \mathbf{u} + P^{\text{eq}} \mathbf{u} \right) = 0 \end{cases} \quad \text{with } P^{\text{eq}} \stackrel{\text{def}}{=} \frac{S_\tau^{\text{eq}}}{S_\varepsilon^{\text{eq}}}.$$

PROPERTIES

If $\tau_1^* \neq \tau_2^*$ and $\varepsilon_1^* \neq \varepsilon_2^*$ (first order phase transition) then

$$\textcircled{1} c(\mathbf{w}) > 0, \quad \textcircled{2} s_{\tau\varepsilon}^{\text{eq}}(\mathbf{w}) > 0$$

- ① Euler system: strict hyperbolicity (\neq p-system),
- ② Riemann problem: multitude of entropy (Lax) solutions [R. MENIKOFF, B. J. PLOHR], uniqueness of Liu solution.

Dynamic Liquid-Vapor Phase Change



EULER SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \right) + \operatorname{div} \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \mathbf{u} + P^{\text{eq}} \mathbf{u} \right) = 0 \end{cases} \quad \text{with } P^{\text{eq}} \stackrel{\text{def}}{=} \frac{S_\tau^{\text{eq}}}{S_\varepsilon^{\text{eq}}}.$$

PROPERTIES

If $\tau_1^* \neq \tau_2^*$ and $\varepsilon_1^* \neq \varepsilon_2^*$ (first order phase transition) then

$$\textcircled{1} c(\mathbf{w}) > 0, \quad \textcircled{2} s_{\tau\varepsilon}^{\text{eq}}(\mathbf{w}) > 0$$

- ① Euler system: **strict hyperbolicity** (\neq p-system), 
- ② Riemann problem: multitude of entropy (Lax) solutions [R. MENIKOFF, B. J. PLOHR], uniqueness of Liu solution. 

Dynamic Liquid-Vapor Phase Change

EULER SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \right) + \operatorname{div} \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \mathbf{u} + P^{\text{eq}} \mathbf{u} \right) = 0 \end{cases} \quad \text{with } P^{\text{eq}} \stackrel{\text{def}}{=} \frac{S_\tau^{\text{eq}}}{S_\varepsilon^{\text{eq}}}.$$

PROPERTIES

If $\tau_1^* \neq \tau_2^*$ and $\varepsilon_1^* \neq \varepsilon_2^*$ (first order phase transition) then

$$\textcircled{1} c(\mathbf{w}) > 0, \quad \textcircled{2} s_{\tau\varepsilon}^{\text{eq}}(\mathbf{w}) > 0$$

- ① Euler system: strict hyperbolicity (\neq p-system), ▶
- ② Riemann problem: multitude of entropy (Lax) solutions [R. MENIKOFF, B. J. PLOHR], uniqueness of Liu solution. ▶

Outline

1 Context

2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

How to simulate the “Liu solution”

- Exact Riemann Solver
like [A. VOSS] for Van der Waals EOS
- Viscous Solver (the Liu solution is the only solution that has a viscous profile)
like [S. JAOUEN] for Perfect Gas EOS with $c_{V_{liq}} = c_{V_{vap}}$
- Solver(s) based on **Relaxation Approach**
[F. COQUEL, B. PERTHAME],
[Th. BARBERON, Ph. HELLUY],
[Ph. HELLUY, N. SEGUIN],
[F. COQUEL, F. CARO, D. JAMET, S. KOKH],
[R. ABGRALL, R. SAUREL],
[V. GUILLEMAUD, J.-M. HÉRARD, S. KOKH],
...

Relaxation Approach

$$\partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

EQUILIBRIUM SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = \mathbf{0} \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P^{\text{eq}}) \mathbf{u}) = 0 \end{cases}$$

$$P^{\text{eq}}(\rho, \varepsilon) = \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}, \quad e \stackrel{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

HOW TO BUILD THE AUGMENTED SYSTEM

1 Lagrangian:

$$\mathcal{L}(\rho, \mathbf{u}, \sigma, y, z, \psi) \stackrel{\text{def}}{=} \rho \left(\frac{|\mathbf{u}|^2}{2} - \varepsilon(\rho, \sigma, y, z, \psi) \right)$$

Action:

$$\mathcal{A}(\mathbf{v}) \stackrel{\text{def}}{=} \int_{t_1}^{t_2} \int_{\hat{\Omega}(t; \mathbf{v})} \mathcal{L}(\hat{\rho}, \hat{\rho} \mathbf{u}, \hat{s}, \hat{y}, \hat{z}, \hat{\psi})(\hat{\mathbf{x}}, t; \mathbf{v}) d\hat{\mathbf{x}} dt$$

Minimization of the Action: $\frac{d\mathcal{A}}{d\mathbf{v}}(\mathbf{v} = 0) = 0$

2 Energy: $\varepsilon \stackrel{\text{def}}{=} \sum_{\alpha} y_{\alpha} \varepsilon_{\alpha} \left(\frac{z_{\alpha}}{y_{\alpha}} \frac{1}{\rho}, \frac{\psi_{\alpha}}{y_{\alpha}} \sigma \right)$

3 Positive Entropy Production: $D_t \sigma \geq 0$

EQUILIBRIUM SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P^{\text{eq}}) \mathbf{u}) = 0 \end{cases}$$

$$P^{\text{eq}}(\rho, \varepsilon) = \frac{S_{\varepsilon}^{\text{eq}}}{S_{\rho}^{\text{eq}}}, \quad e \stackrel{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

AUGMENTED SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P)\mathbf{u}) = 0 \end{cases}$$

$$P(\rho, \varepsilon, z, y, \psi) = \frac{\sigma_\tau}{\sigma_\varepsilon}$$

EQUILIBRIUM SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P^{\text{eq}})\mathbf{u}) = 0 \end{cases}$$

$$P^{\text{eq}}(\rho, \varepsilon) = \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}, \quad e \stackrel{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

AUGMENTED SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P)\mathbf{u}) = 0 \end{cases}$$

In the interface

$$\begin{cases} \partial_t z + \mathbf{u} \cdot \operatorname{grad} z = \\ \partial_t y + \mathbf{u} \cdot \operatorname{grad} y = \\ \partial_t \psi + \mathbf{u} \cdot \operatorname{grad} \psi = \end{cases}$$

$$P(\rho, \varepsilon, z, y, \psi) = \frac{\sigma_\tau}{\sigma_\varepsilon}$$

EQUILIBRIUM SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P^{\text{eq}})\mathbf{u}) = 0 \end{cases}$$

$$P^{\text{eq}}(\rho, \varepsilon) = \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}, \quad e \stackrel{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

AUGMENTED SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P) \mathbf{u}) = 0 \end{cases}$$

In the interface

$$\begin{cases} \partial_t z + \mathbf{u} \cdot \operatorname{grad} z = \frac{1}{\mu_z} \left(\frac{P_2}{T_2} - \frac{P_1}{T_1} \right) \\ \partial_t y + \mathbf{u} \cdot \operatorname{grad} y = \frac{1}{\mu_y} \left(\frac{g_1}{T_1} - \frac{g_2}{T_2} \right) \frac{1}{\rho} \\ \partial_t \psi + \mathbf{u} \cdot \operatorname{grad} \psi = \frac{1}{\mu_\psi} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \varepsilon \end{cases}$$

$$P(\rho, \varepsilon, z, y, \psi) = \frac{\sigma_\tau}{\sigma_\varepsilon}$$

Formally
 $\mu_j \rightarrow 0$

EQUILIBRIUM SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P^{\text{eq}}) \mathbf{u}) = 0 \end{cases}$$

$$P^{\text{eq}}(\rho, \varepsilon) = \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}, \quad e \stackrel{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

Relaxation Approach

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu} \mathbf{R}(\mathbf{V}) \quad \xrightarrow[\mu \rightarrow 0]{\text{Formally}} \quad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

AUGMENTED SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P) \mathbf{u}) = 0 \end{cases}$$

In the interface

$$\begin{cases} \partial_t z + \mathbf{u} \cdot \operatorname{grad} z = \frac{1}{\mu_z} \left(\frac{P_2}{T_2} - \frac{P_1}{T_1} \right) \\ \partial_t y + \mathbf{u} \cdot \operatorname{grad} y = \frac{1}{\mu_y} \left(\frac{g_1}{T_1} - \frac{g_2}{T_2} \right) \frac{1}{\rho} \\ \partial_t \psi + \mathbf{u} \cdot \operatorname{grad} \psi = \frac{1}{\mu_\psi} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \varepsilon \end{cases}$$

$$P(\rho, \varepsilon, z, y, \psi) = \frac{\sigma_\tau}{\sigma_\varepsilon}$$

Formally
 $\mu_j \rightarrow 0$

EQUILIBRIUM SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}} \mathbb{I}) = 0 \\ \partial_t(\rho e) + \operatorname{div}((\rho e + P^{\text{eq}}) \mathbf{u}) = 0 \end{cases}$$

$$P^{\text{eq}}(\rho, \varepsilon) = \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}, \quad e \stackrel{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

NOTE: we can replace an EDP by an algebraic closure, for example

~~$$\partial_t \psi + \mathbf{u} \cdot \operatorname{grad} \psi = \frac{1}{\mu_\psi} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \varepsilon \rightsquigarrow T_1 = T_2.$$~~

Numerical Scheme

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu} \mathbf{R}(\mathbf{V})$$

$$\mathbf{V}_i^n \longrightarrow \mathbf{V}_i^{n+1}$$

$$\textcircled{1} \mu_j \rightarrow +\infty$$



$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V})$$

$$\textcircled{2} \mu_j \rightarrow 0$$



$$\mathbf{R}(\mathbf{V}) = 0$$

Aug. System: 5-eq. iso-T
 Num. Scheme: op. splitting
 $\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = 0$ [G. ALLAIRE and all.]
 $\mathbf{S}(\mathbf{V}) = \mathbf{S}_{\text{at}}(\mathbf{V}) + \mathbf{S}_{\text{trans}}(\mathbf{V}) + \mathbf{S}_{\text{p}}(\mathbf{V})$
 $\partial_t \mathbf{V} - \mathbf{S}_{\text{at}}(\mathbf{V})$ [J. U. BRACKBILL and all.]
 $\partial_t \mathbf{V} - \mathbf{S}_{\text{trans}}(\mathbf{V})$ 2D implicit
 $\partial_t \mathbf{V} - \mathbf{S}_{\text{p}}(\mathbf{V})$ Euler

update fractions
 (y, z, ψ)
 solving the
 Phase Change
 Equation

Numerical Scheme

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu} \mathbf{R}(\mathbf{V})$$

$$\mathbf{V}_i^n \longrightarrow \mathbf{V}_i^{n+1}$$

$$\textcircled{1} \mu_j \rightarrow +\infty$$



$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V})$$

$$\textcircled{2} \mu_j \rightarrow 0$$



$$\mathbf{R}(\mathbf{V}) = \mathbf{0}$$

Aug. System: 5-eq. iso-T
 Num. Scheme: op. splitting
 $\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{0}$ [G. ALLAIRE and all.]

$$\mathbf{S}(\mathbf{V}) = \mathbf{S}_{\text{st}}(\mathbf{V}) + \mathbf{S}_{\text{heat}}(\mathbf{V}) + \mathbf{S}_{\text{g}}(\mathbf{V})$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{st}}(\mathbf{V}) \quad [\text{J. U. BRACKBILL and all.}]$$

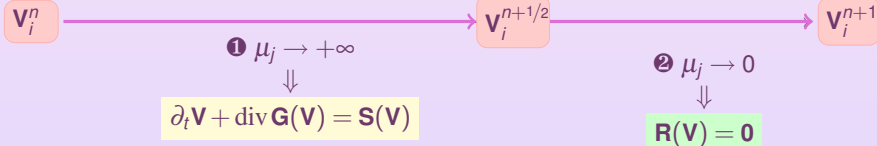
$$\partial_t \mathbf{V} = \mathbf{S}_{\text{heat}}(\mathbf{V}) \quad \text{2D implicit}$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{g}}(\mathbf{V}) \quad \text{Euler}$$

update fractions
 (y, z, ψ)
 solving the
 Phase Change
 Equation

Numerical Scheme

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu} \mathbf{R}(\mathbf{V})$$



Aug. System: 5-eq. iso-T
 Num. Scheme: op. splitting
 $\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = 0$ [G. ALLAIRE and all.]

$$\mathbf{S}(\mathbf{V}) = \mathbf{S}_{\text{st}}(\mathbf{V}) + \mathbf{S}_{\text{heat}}(\mathbf{V}) + \mathbf{S}_{\text{g}}(\mathbf{V})$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{st}}(\mathbf{V}) \quad [\text{J. U. BRACKBILL and all.}]$$

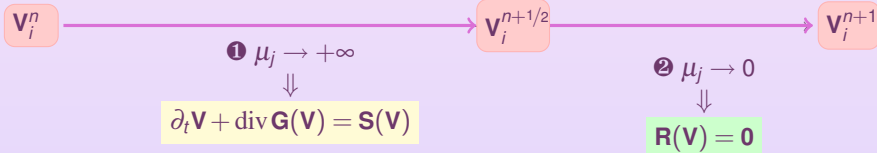
$$\partial_t \mathbf{V} = \mathbf{S}_{\text{heat}}(\mathbf{V}) \quad \text{2D implicit}$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{g}}(\mathbf{V}) \quad \text{Euler}$$

update fractions
 (y, z, ψ)
 solving the
 Phase Change
 Equation

Numerical Scheme

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu} \mathbf{R}(\mathbf{V})$$



Aug. System: 5-eq. iso-T
 Num. Scheme: op. splitting
 $\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{0}$ [G. ALLAIRE and all.]

$$\mathbf{S}(\mathbf{V}) = \mathbf{S}_{\text{st}}(\mathbf{V}) + \mathbf{S}_{\text{heat}}(\mathbf{V}) + \mathbf{S}_{\text{g}}(\mathbf{V})$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{st}}(\mathbf{V}) \quad \text{[J. U. BRACKBILL and all.]}$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{heat}}(\mathbf{V}) \quad \text{2D implicit}$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{g}}(\mathbf{V}) \quad \text{Euler}$$

update fractions
 (y, z, ψ)
 solving the
 Phase Change
 Equation

Numerical Scheme

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu} \mathbf{R}(\mathbf{V})$$



$$\textcircled{1} \mu_j \rightarrow +\infty$$



$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V})$$

Aug. System: 5-eq. iso-T
 Num. Scheme: op. splitting
 $\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{0}$ [G. ALLAIRE and all.]

$$\mathbf{S}(\mathbf{V}) = \mathbf{S}_{\text{st}}(\mathbf{V}) + \mathbf{S}_{\text{heat}}(\mathbf{V}) + \mathbf{S}_{\text{g}}(\mathbf{V})$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{st}}(\mathbf{V}) \quad \text{[J. U. BRACKBILL and all.]}$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{heat}}(\mathbf{V}) \quad \text{2D implicit}$$

$$\partial_t \mathbf{V} = \mathbf{S}_{\text{g}}(\mathbf{V}) \quad \text{Euler}$$

$$\textcircled{2} \mu_j \rightarrow 0$$



$$\mathbf{R}(\mathbf{V}) = \mathbf{0}$$

update fractions
 (y, z, ψ)
 solving the
 Phase Change
 Equation

Outline

1 Context

2 Model

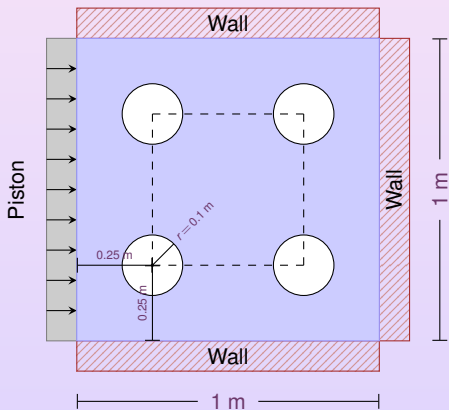
- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

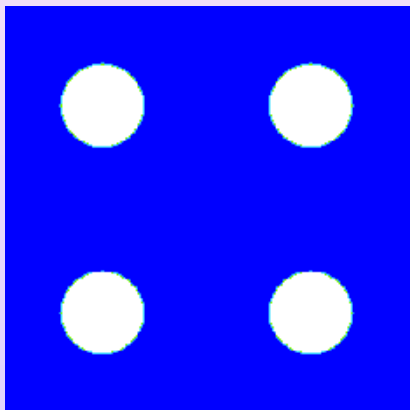
Compression of Vapor Bubbles



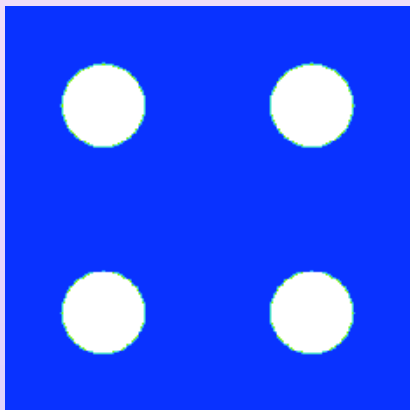
Compression of 4 Vapor Bubbles involving two Stiffened Gases for water and steam. The piston moves towards right at constant speed $u_p = 30$ m/s.

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



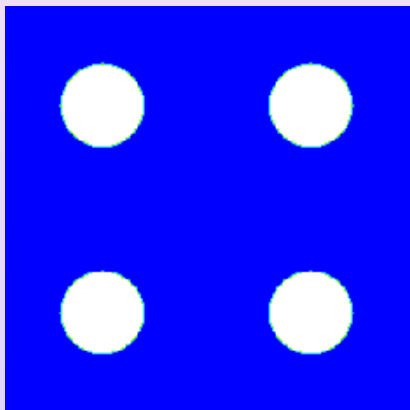
◀ Geometry

▶ Play

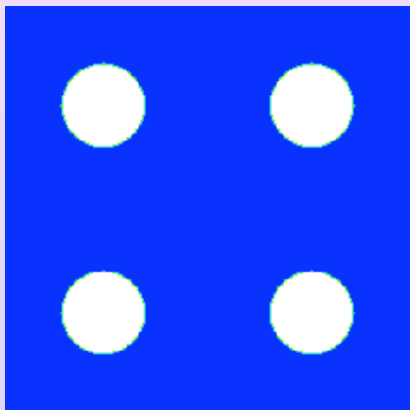
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



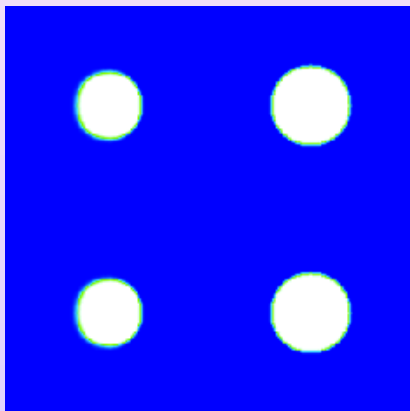
◀ Geometry

▶ Play

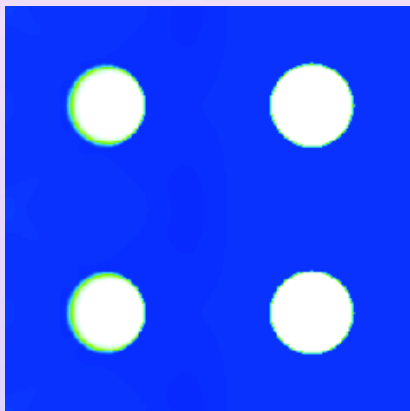
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



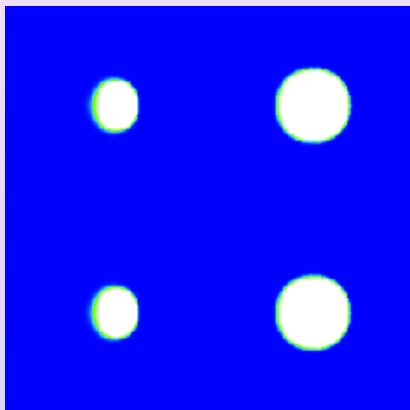
◀ Geometry

▶ Play

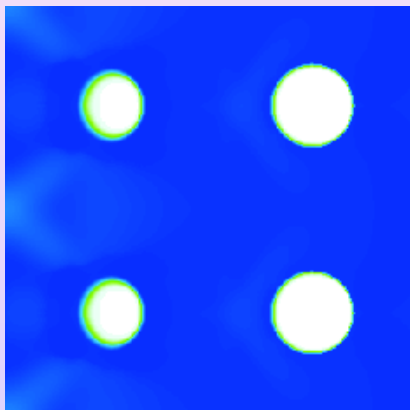
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



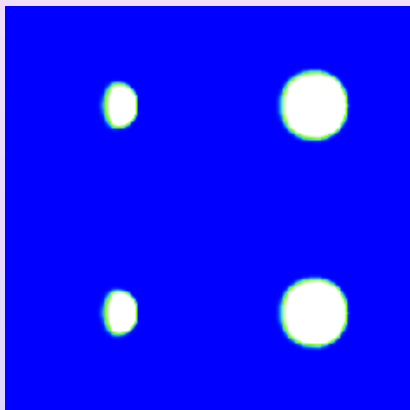
◀ Geometry

▶ Play

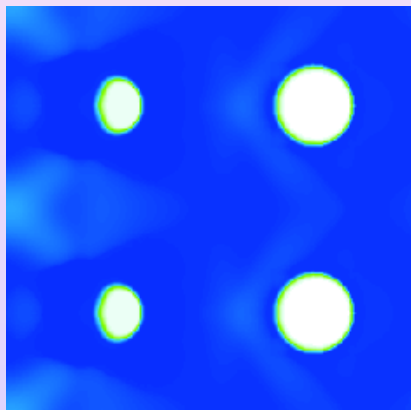
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



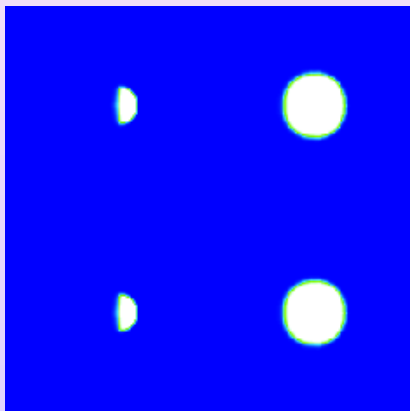
◀ Geometry

▶ Play

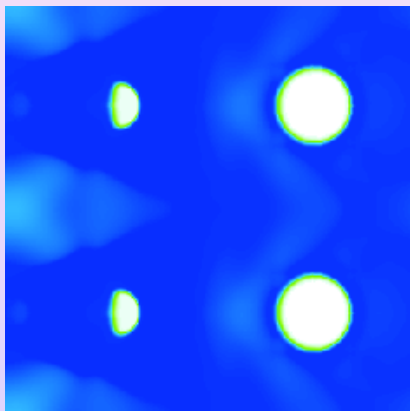
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



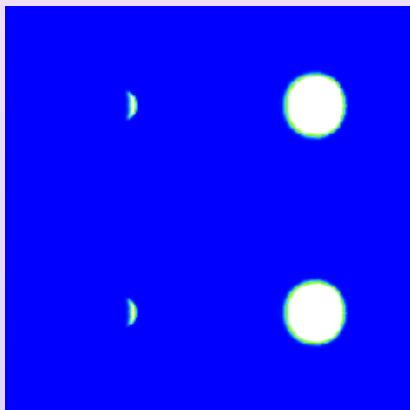
◀ Geometry

▶ Play

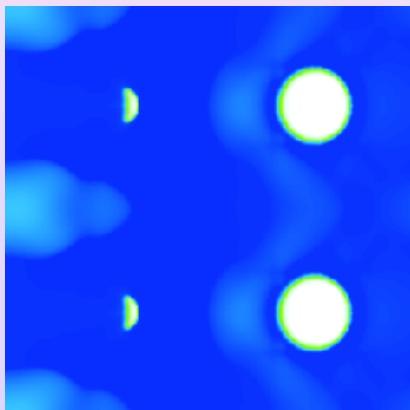
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



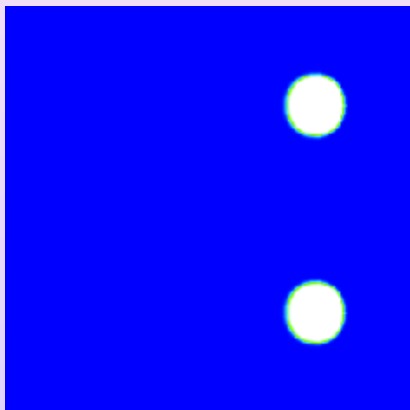
◀ Geometry

▶ Play

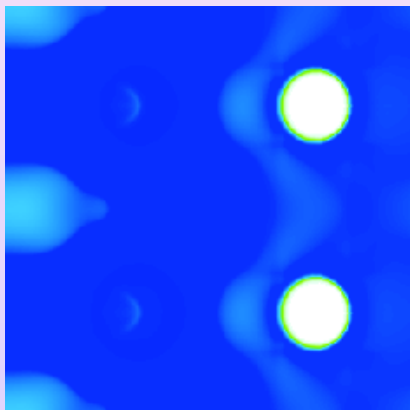
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



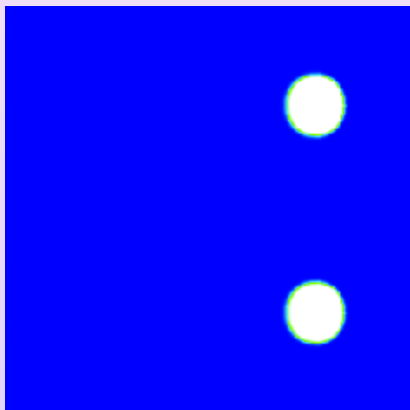
◀ Geometry

▶ Play

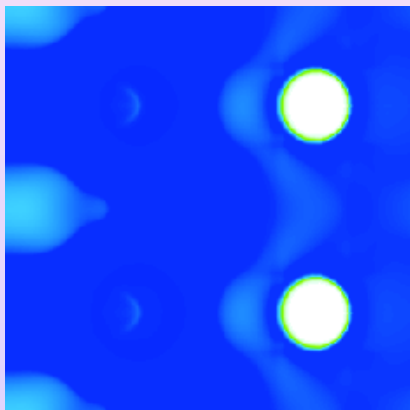
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



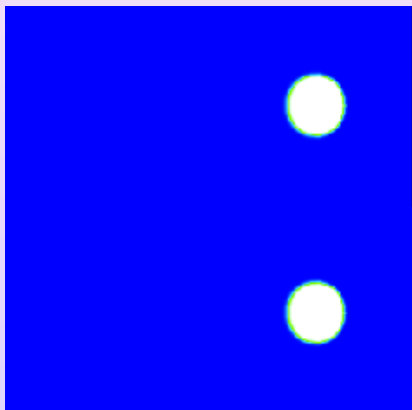
◀ Geometry

▶ Play

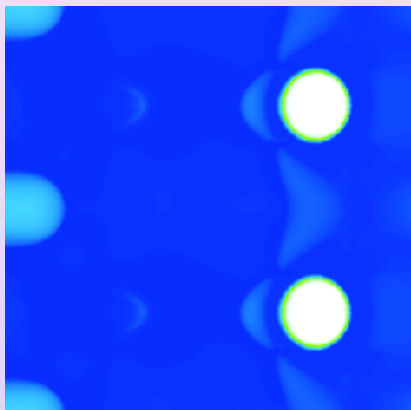
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



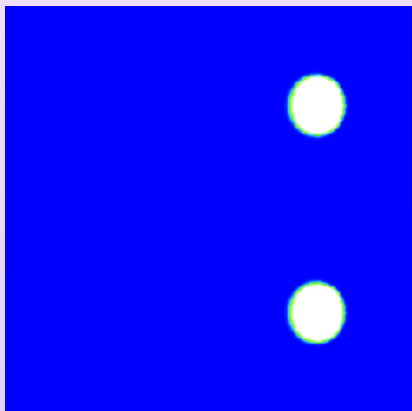
◀ Geometry

▶ Play

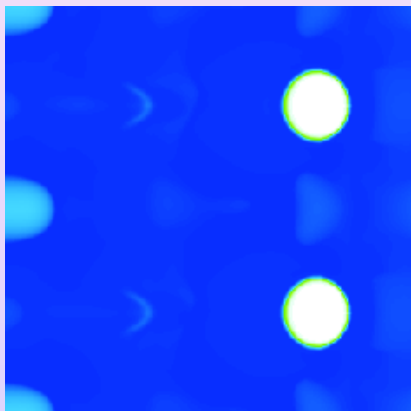
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



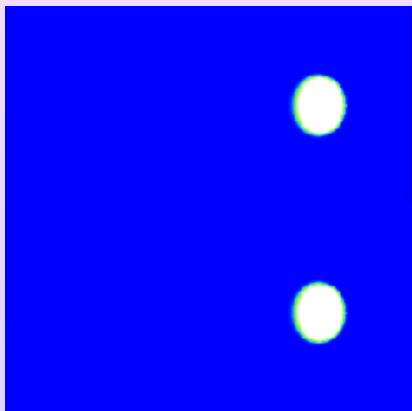
◀ Geometry

▶ Play

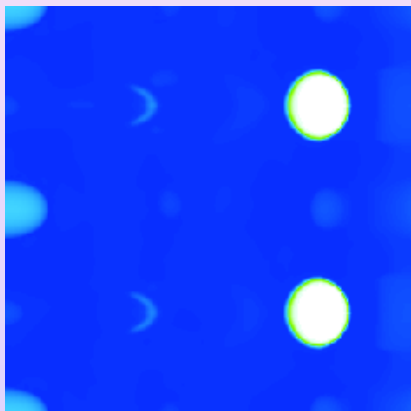
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



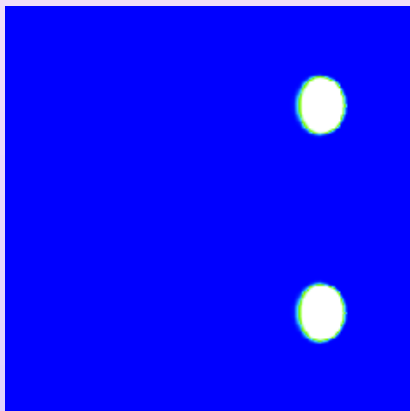
◀ Geometry

▶ Play

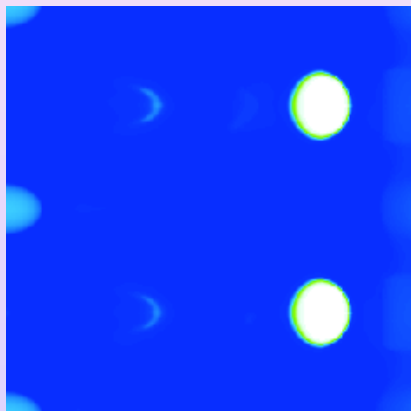
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



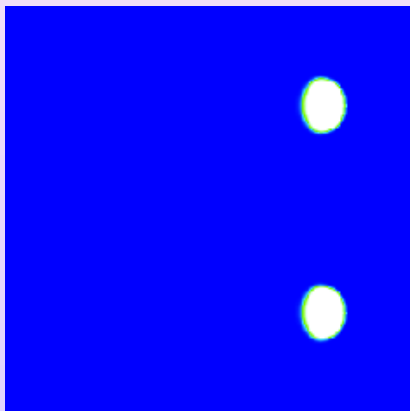
◀ Geometry

▶ Play

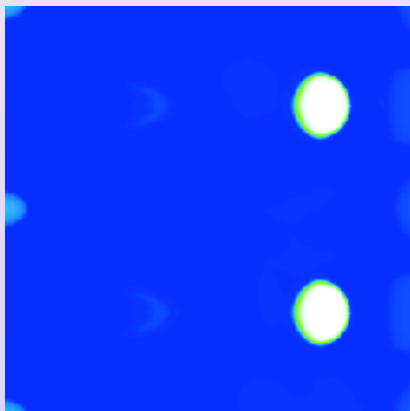
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



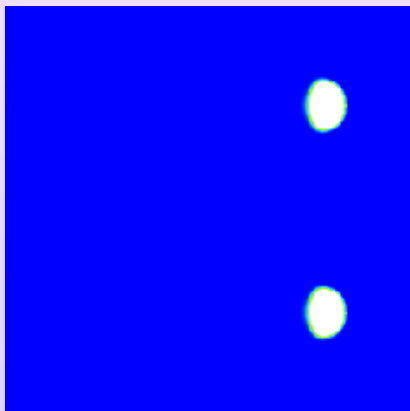
◀ Geometry

▶ Play

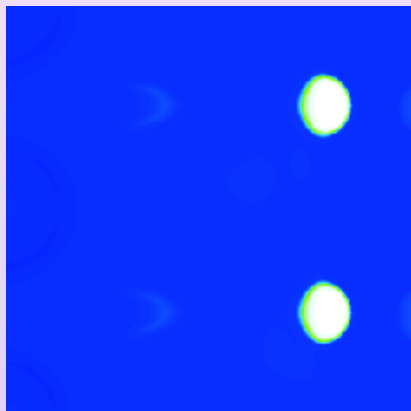
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



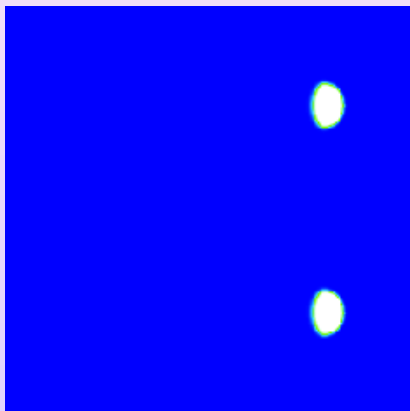
◀ Geometry

▶ Play

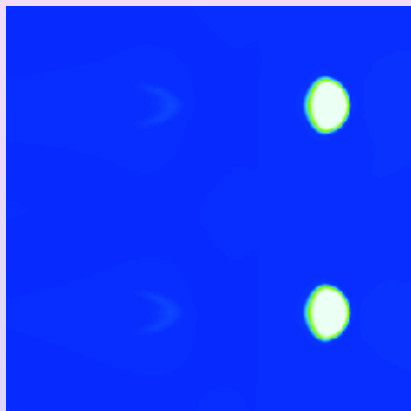
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



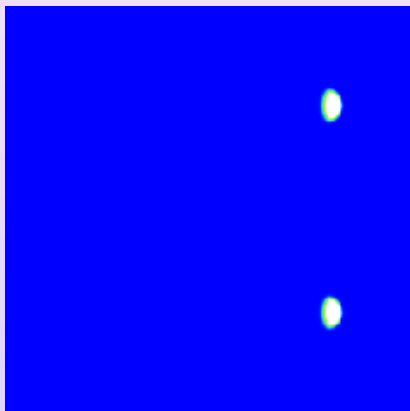
◀ Geometry

▶ Play

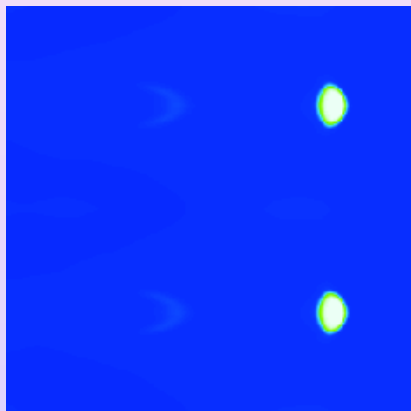
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



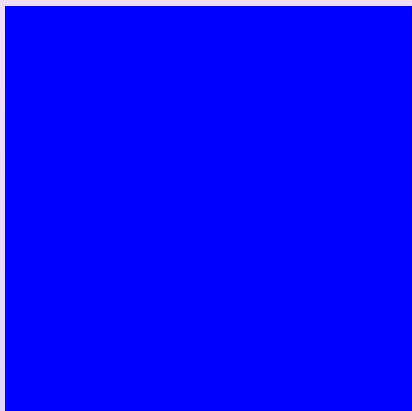
◀ Geometry

▶ Play

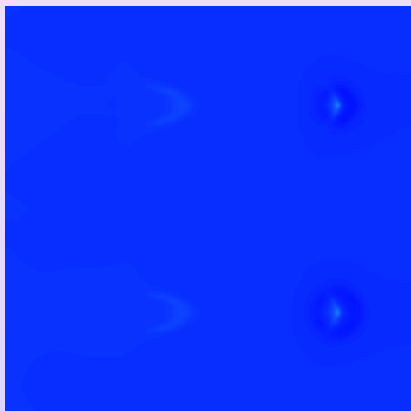
▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



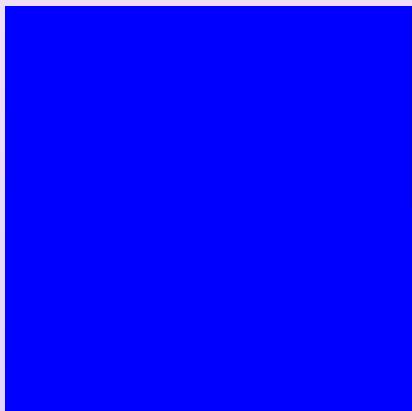
◀ Geometry

▶ Play

▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



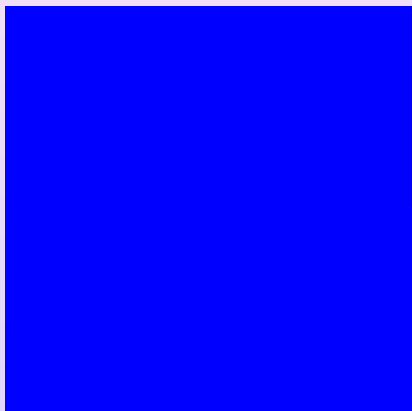
◀ Geometry

▶ Play

▶▶ Skip

Compression of Vapor Bubbles

Mass Fraction y



Density ρ



◀ Geometry

▶ Play

▶▶ Skip

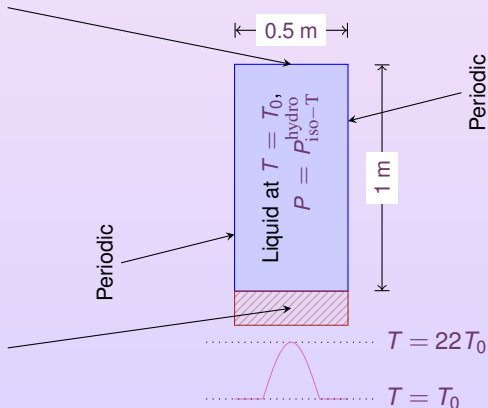
Nucleating Bubble

Pressure and
temperature
imposed

$$P = P^{\text{ref}} > P^{\text{sat}}(T_0),$$

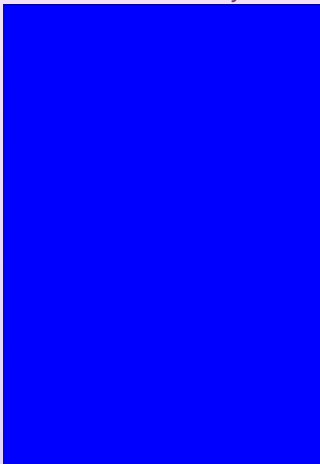
$$T = T_0$$

Wall,
temperature imposed



Nucleating Bubble

Mass Fraction y



Temperature T



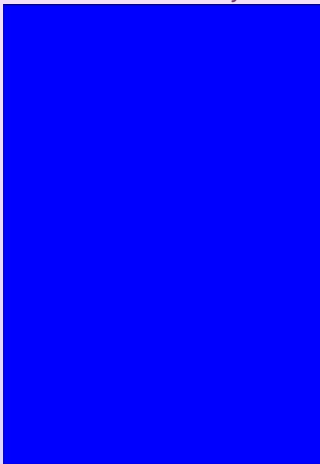
◀ Geometry

▶ Play

▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



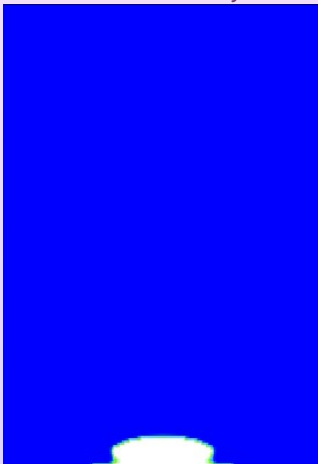
◀ Geometry

▶ Play

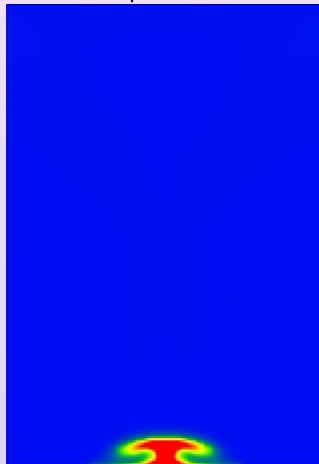
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



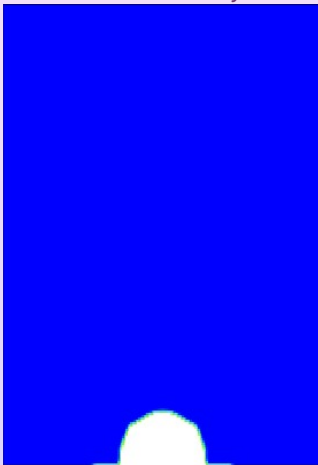
◀ Geometry

▶ Play

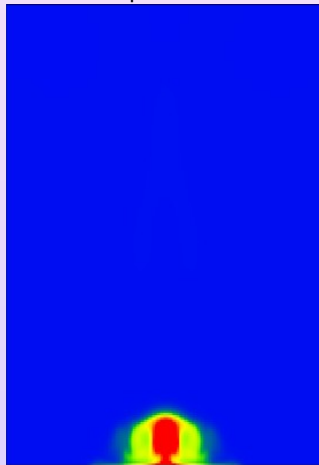
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



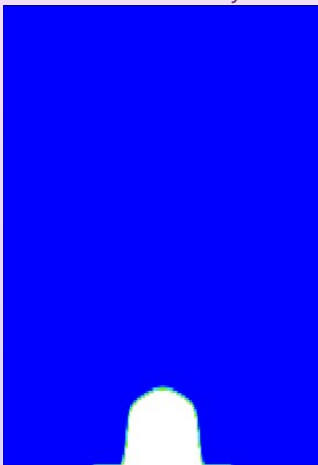
◀ Geometry

▶ Play

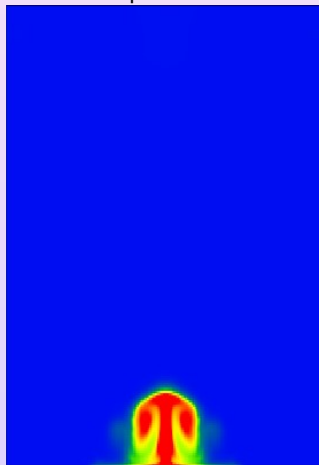
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



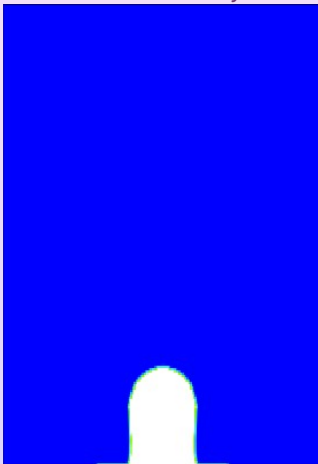
◀ Geometry

▶ Play

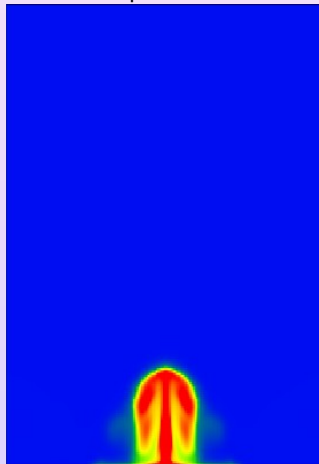
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



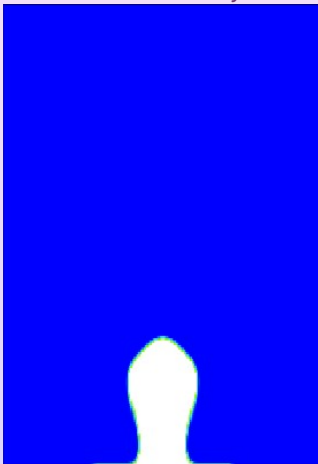
◀ Geometry

▶ Play

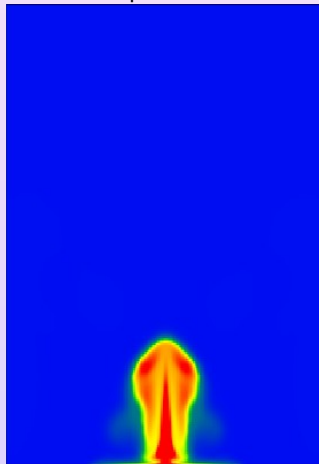
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



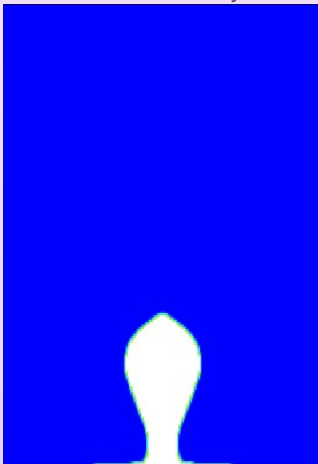
◀ Geometry

▶ Play

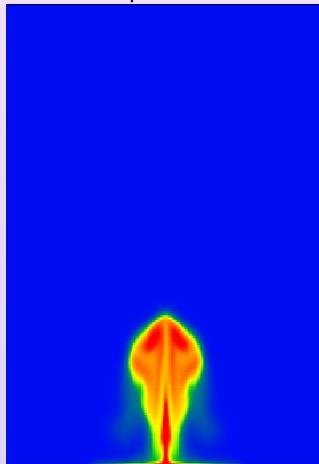
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



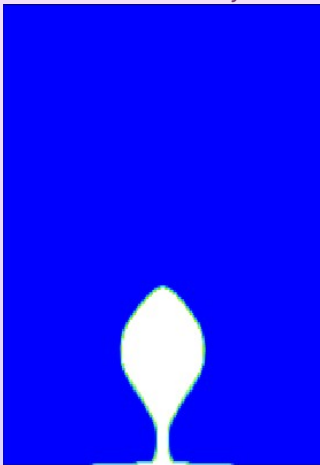
◀ Geometry

▶ Play

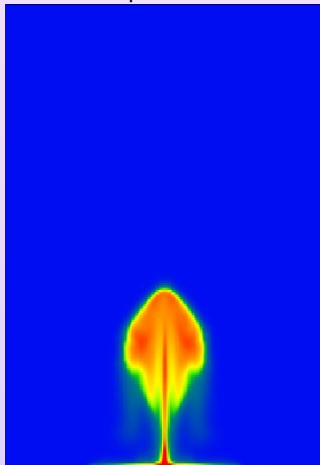
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



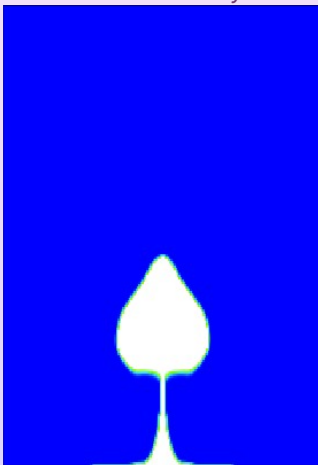
◀ Geometry

▶ Play

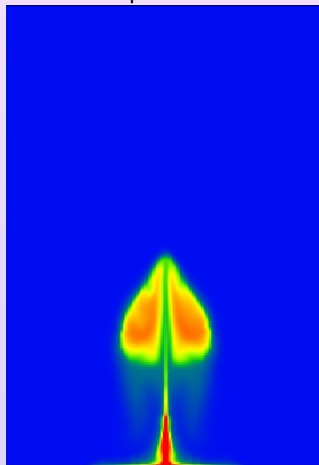
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



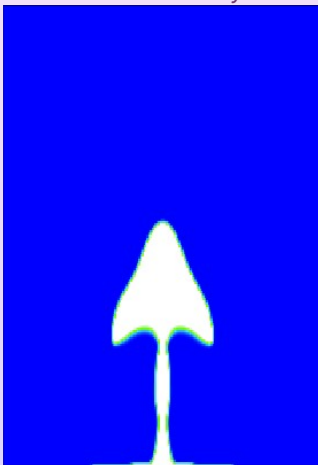
◀ Geometry

▶ Play

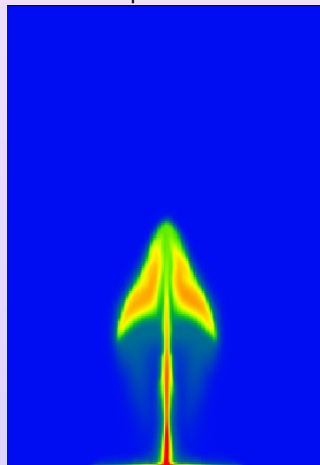
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



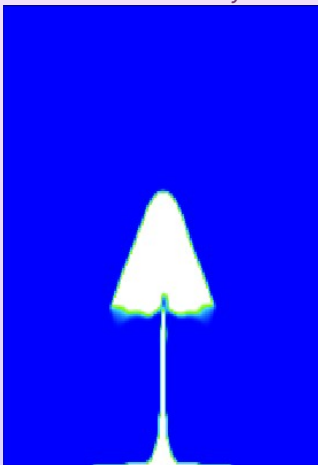
◀ Geometry

▶ Play

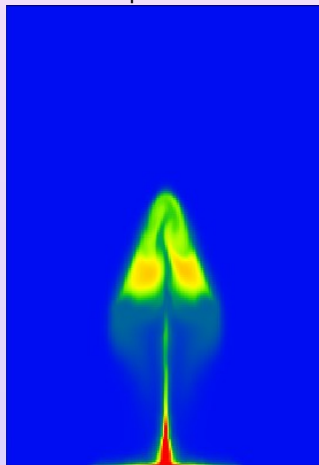
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



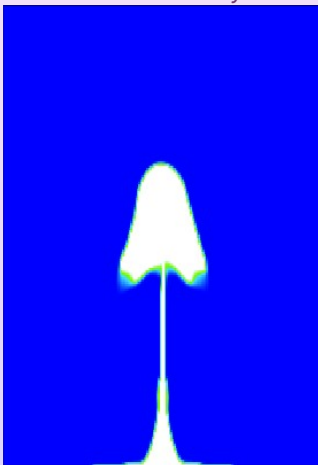
◀ Geometry

▶ Play

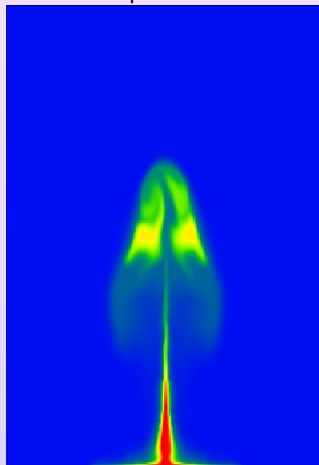
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



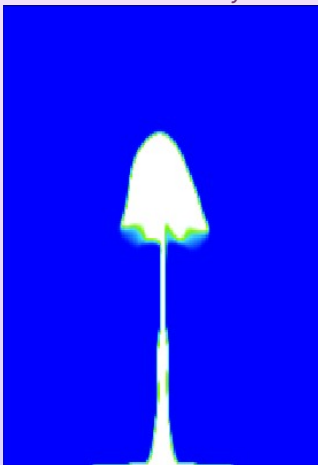
◀ Geometry

▶ Play

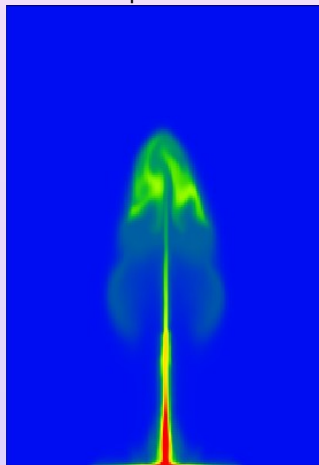
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



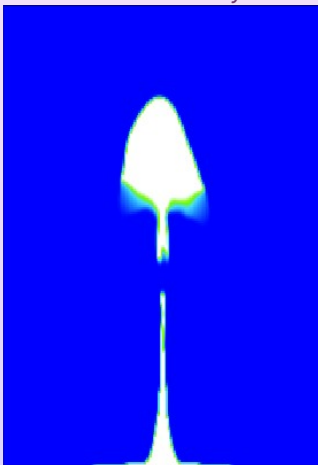
◀ Geometry

▶ Play

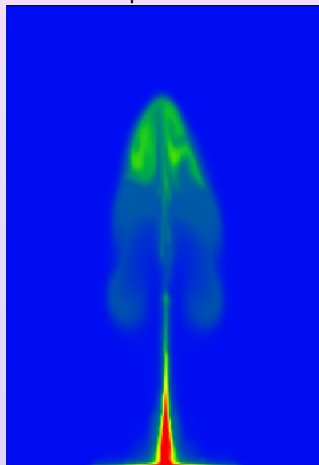
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



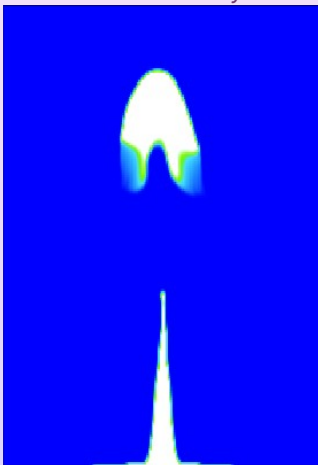
◀ Geometry

▶ Play

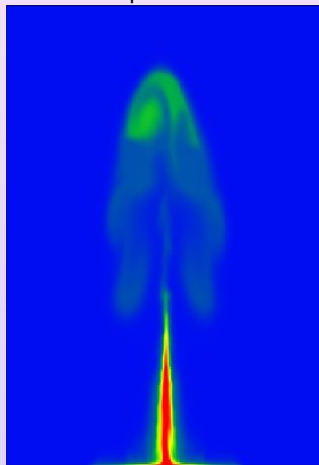
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



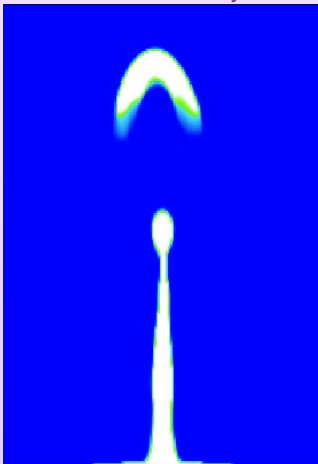
◀ Geometry

▶ Play

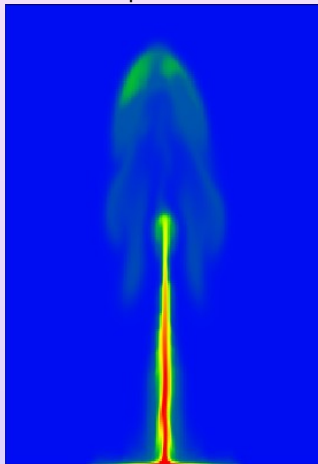
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



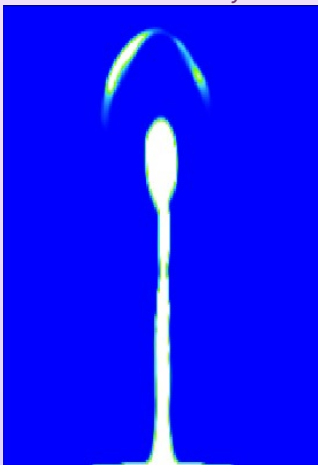
◀ Geometry

▶ Play

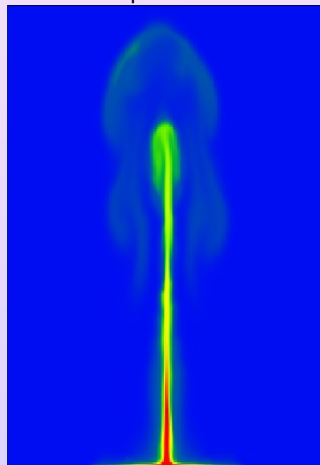
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



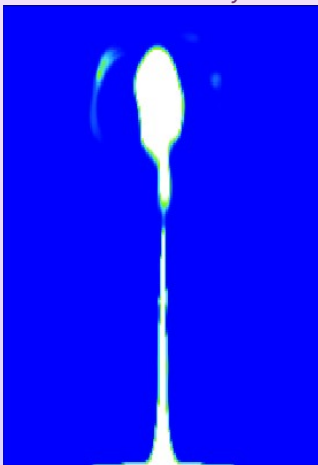
◀ Geometry

▶ Play

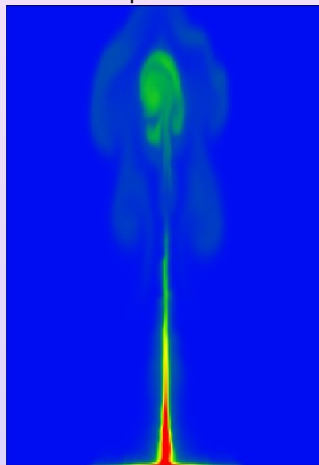
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



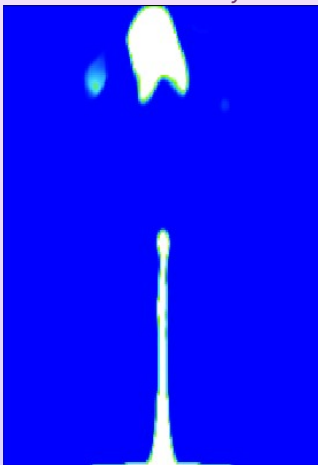
◀ Geometry

▶ Play

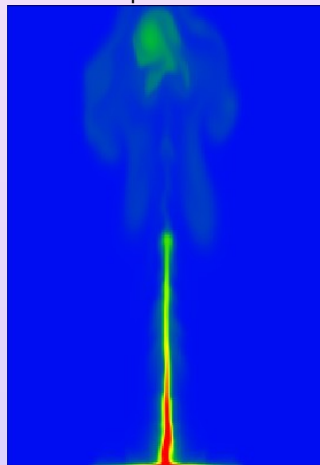
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



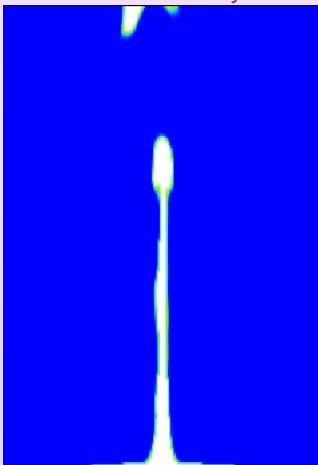
◀ Geometry

▶ Play

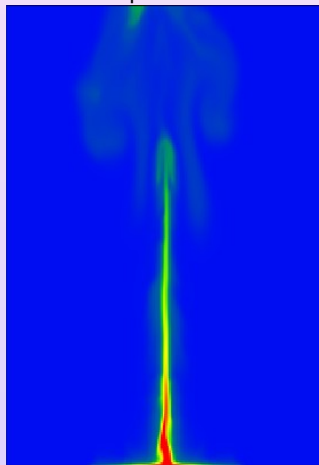
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



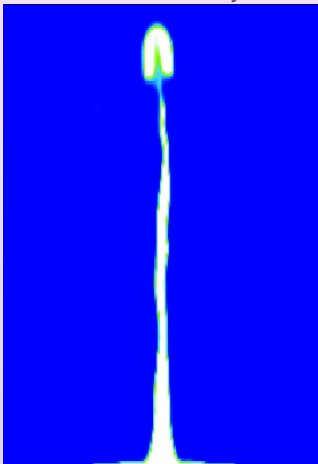
◀ Geometry

▶ Play

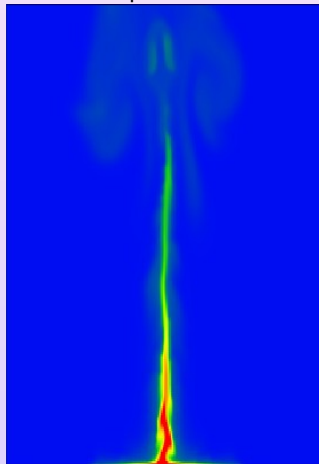
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



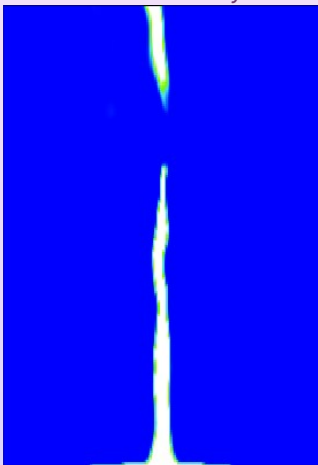
◀ Geometry

▶ Play

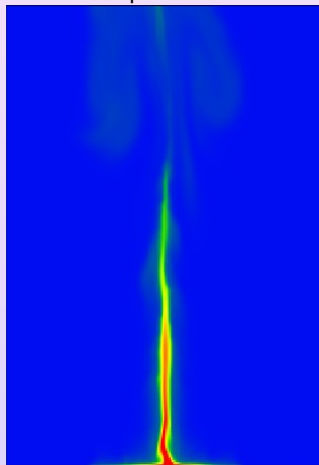
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



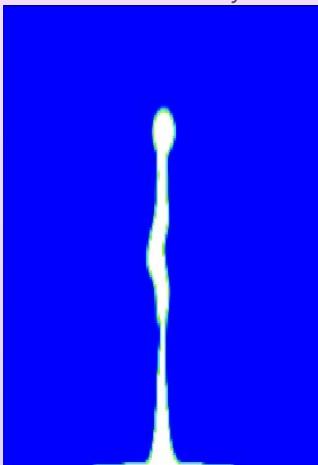
◀ Geometry

▶ Play

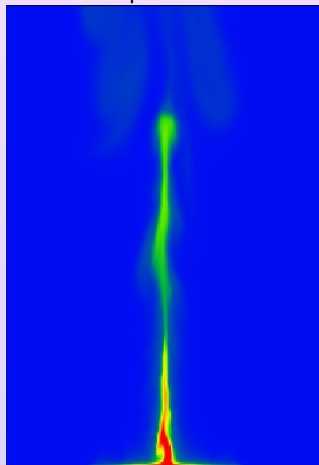
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



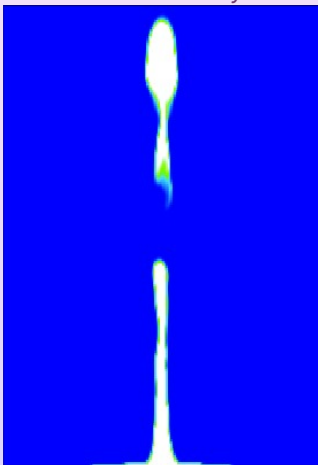
◀ Geometry

▶ Play

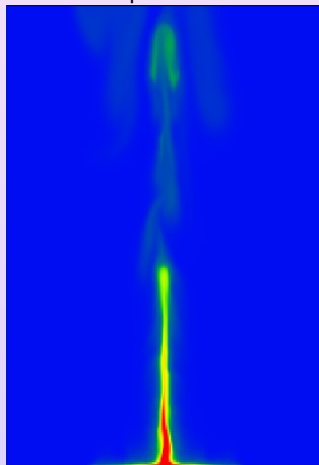
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



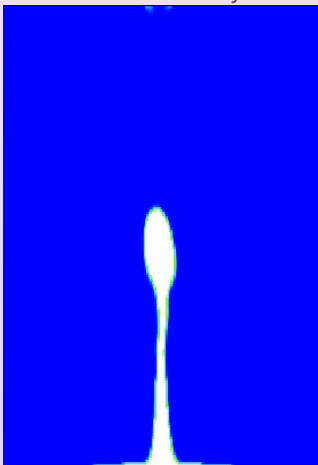
◀ Geometry

▶ Play

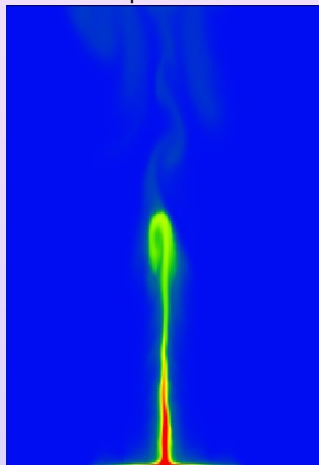
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



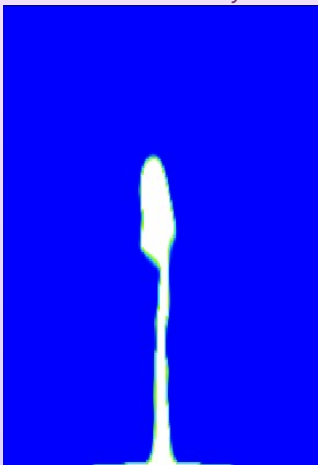
◀ Geometry

▶ Play

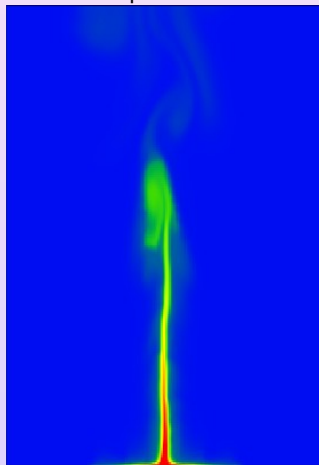
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



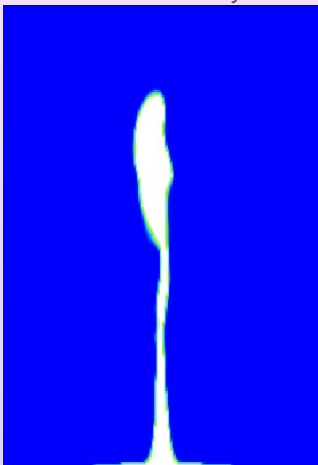
◀ Geometry

▶ Play

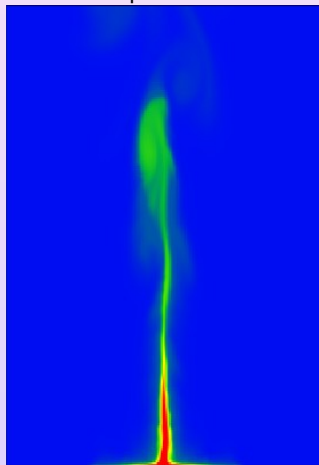
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



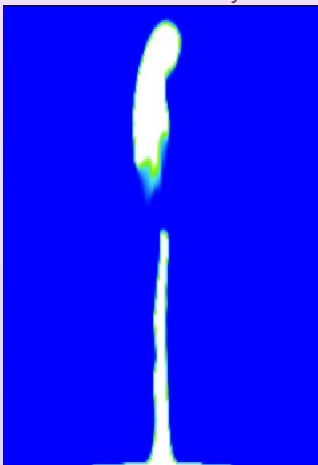
◀ Geometry

▶ Play

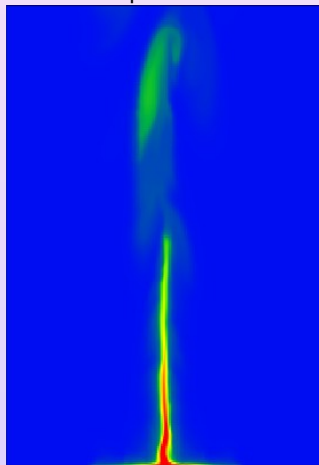
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



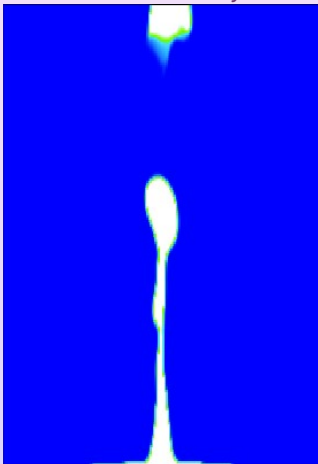
◀ Geometry

▶ Play

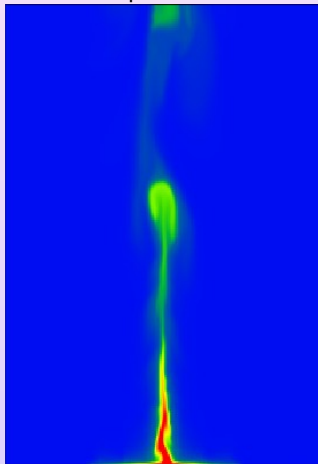
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



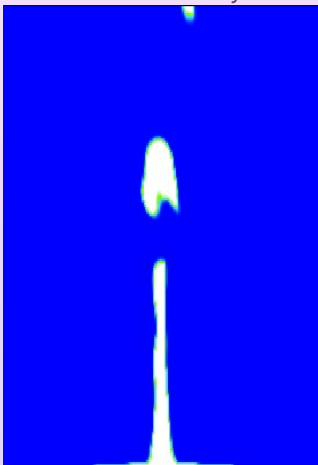
◀ Geometry

▶ Play

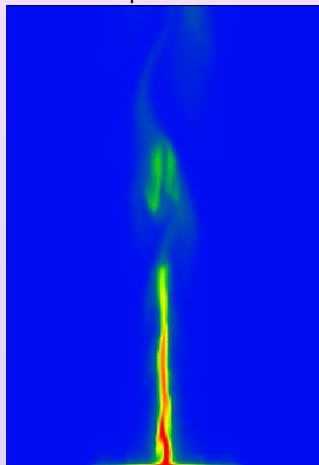
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



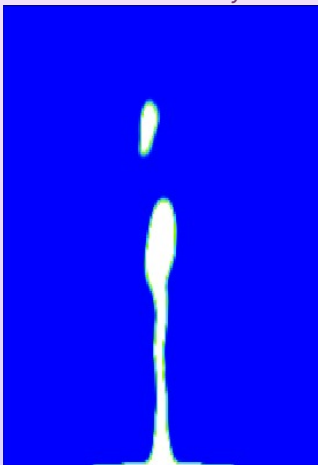
◀ Geometry

▶ Play

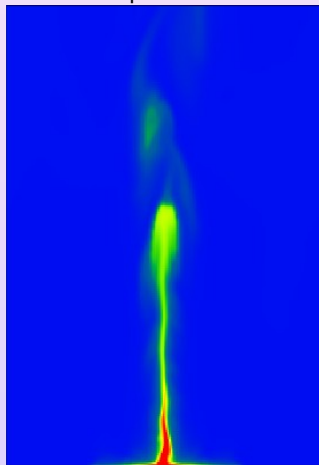
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



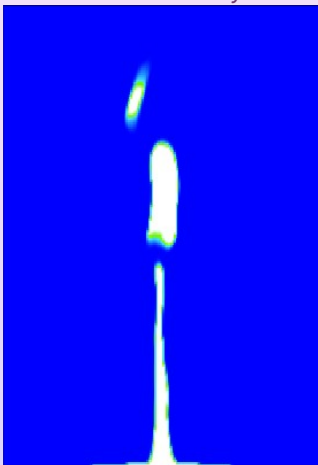
◀ Geometry

▶ Play

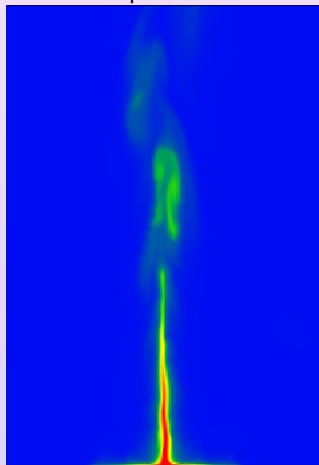
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



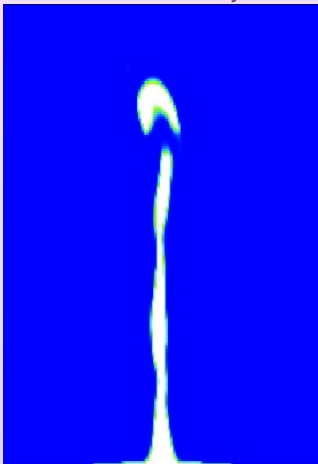
◀ Geometry

▶ Play

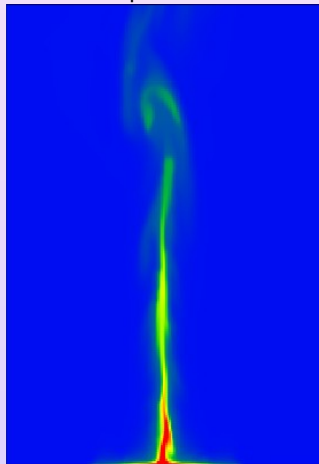
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



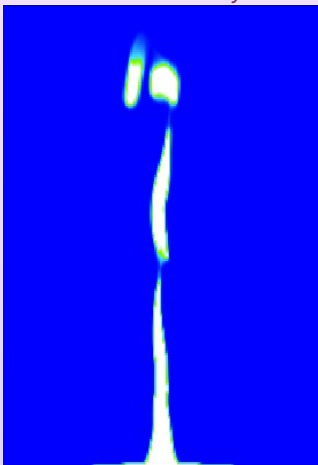
◀ Geometry

▶ Play

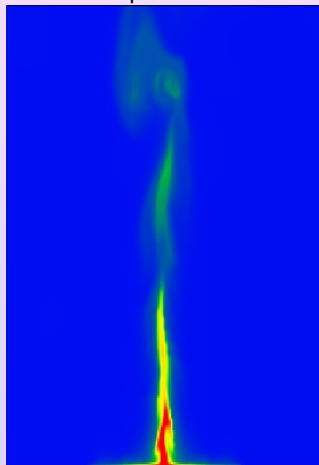
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



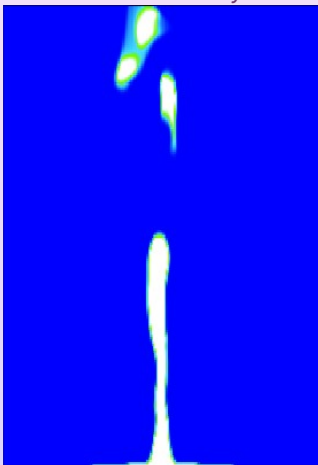
◀ Geometry

▶ Play

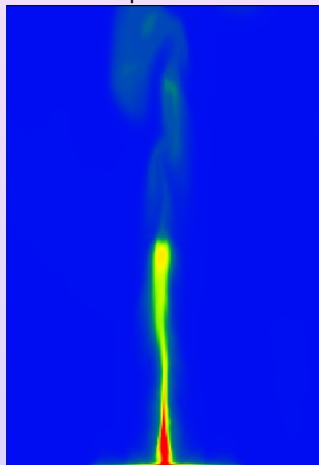
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



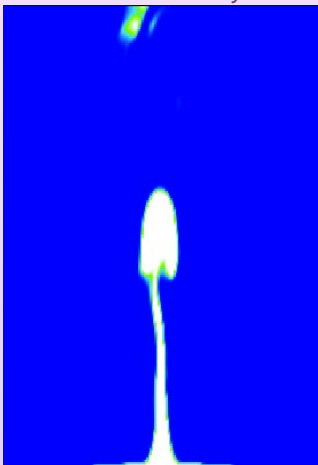
◀ Geometry

▶ Play

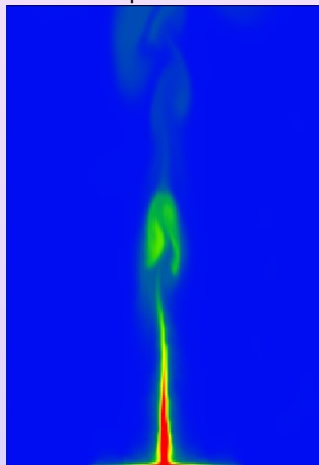
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



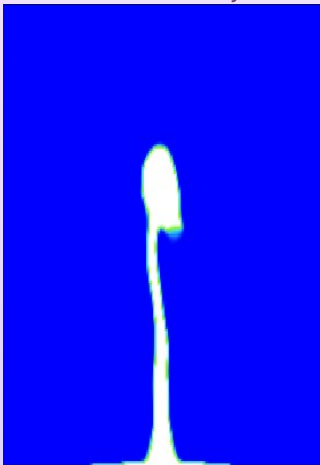
◀ Geometry

▶ Play

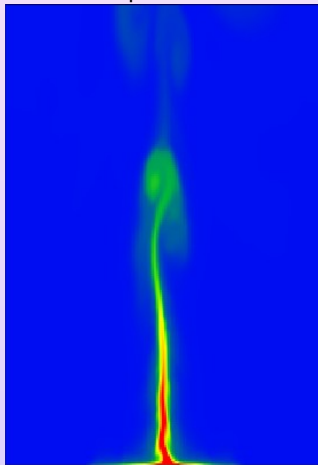
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



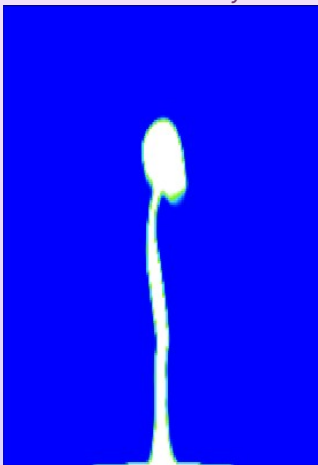
◀ Geometry

▶ Play

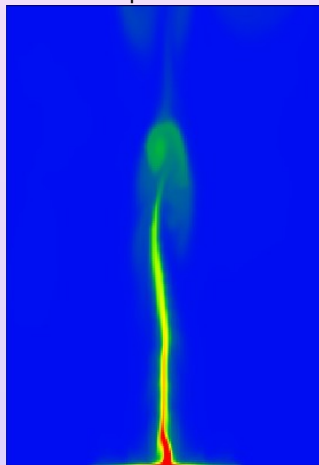
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



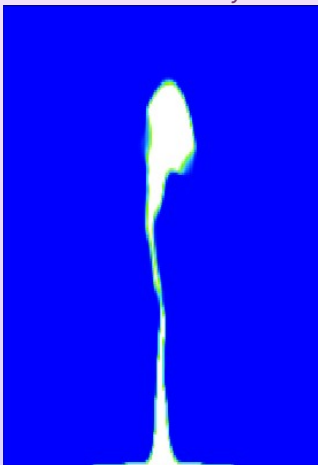
◀ Geometry

▶ Play

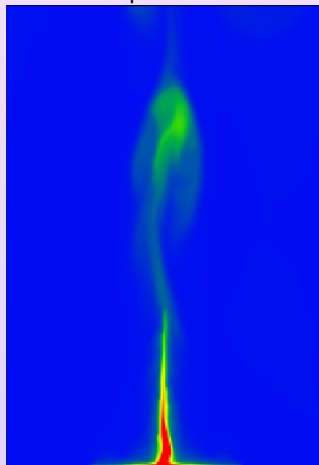
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



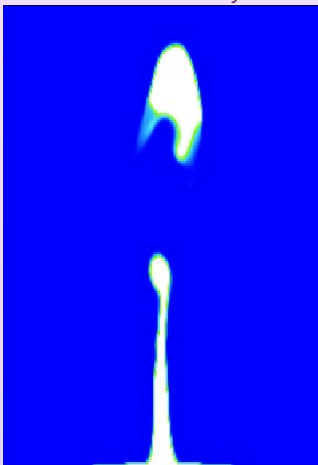
◀ Geometry

▶ Play

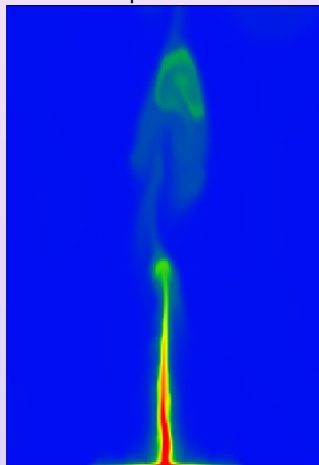
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



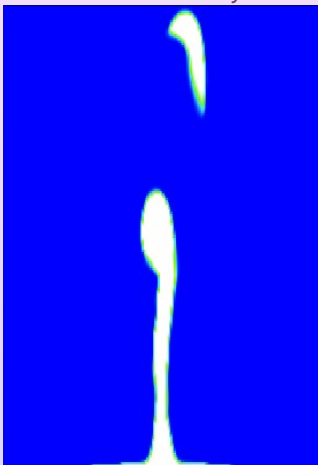
◀ Geometry

▶ Play

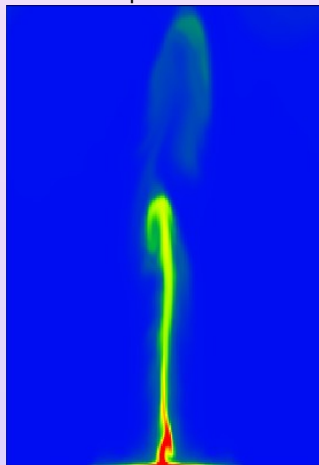
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



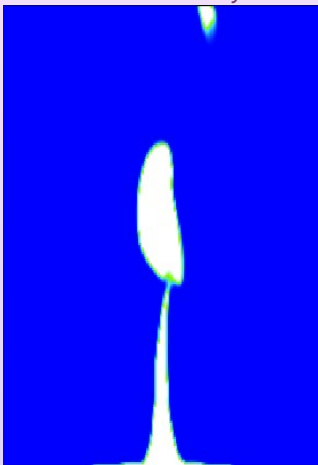
◀ Geometry

▶ Play

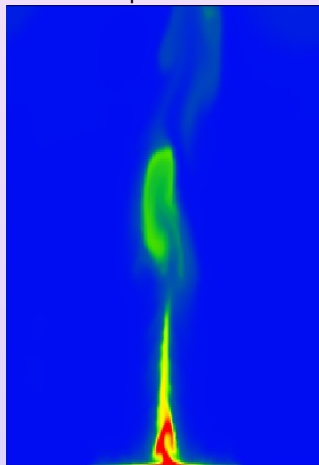
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



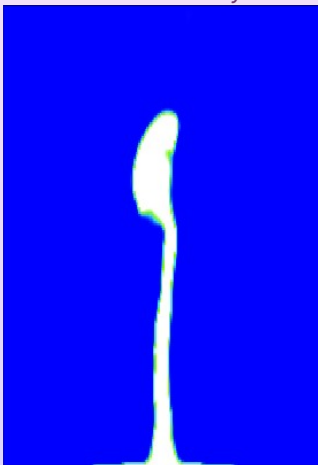
◀ Geometry

▶ Play

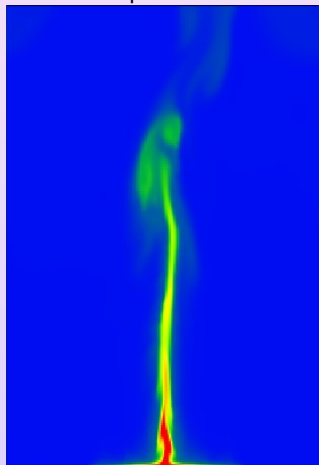
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



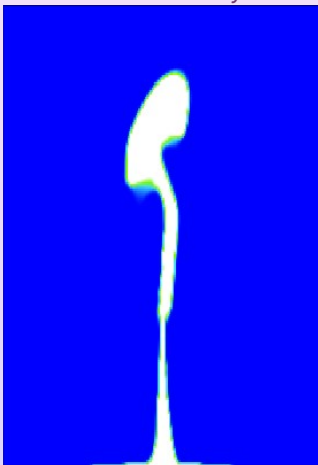
◀ Geometry

▶ Play

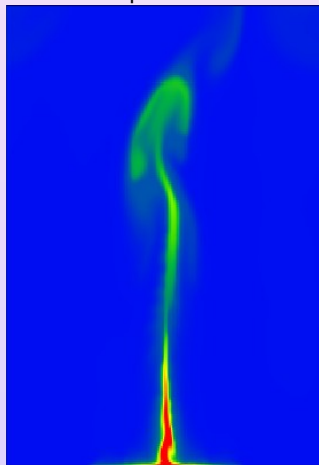
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



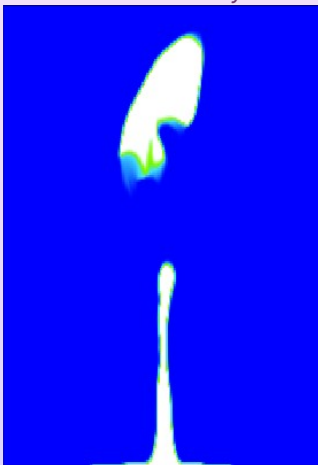
◀ Geometry

▶ Play

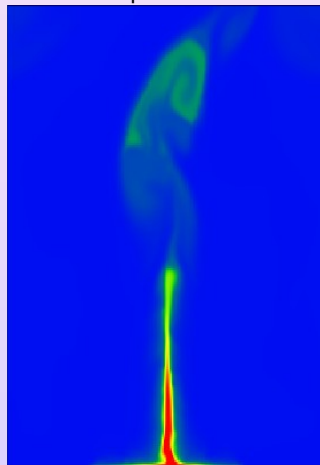
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



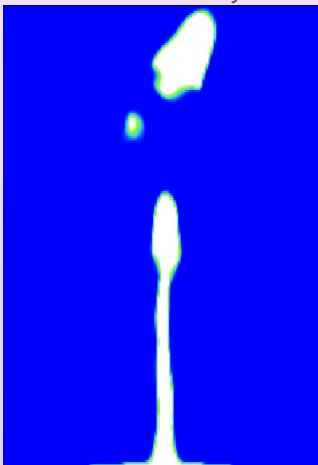
◀ Geometry

▶ Play

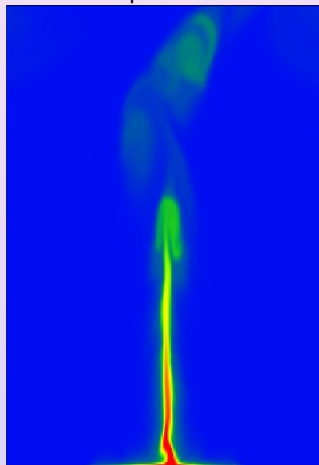
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



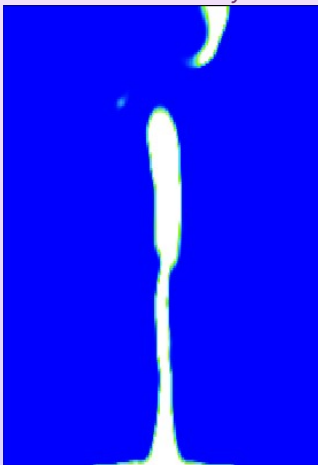
◀ Geometry

▶ Play

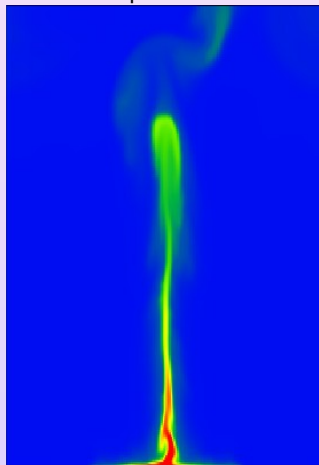
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



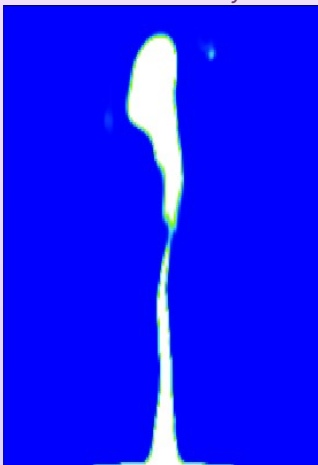
◀ Geometry

▶ Play

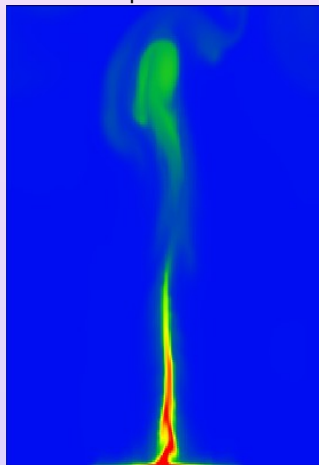
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



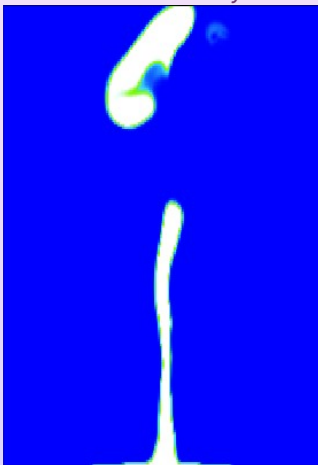
◀ Geometry

▶ Play

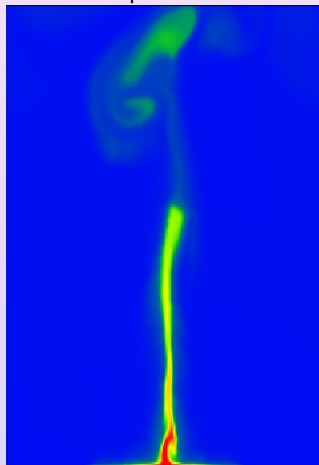
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



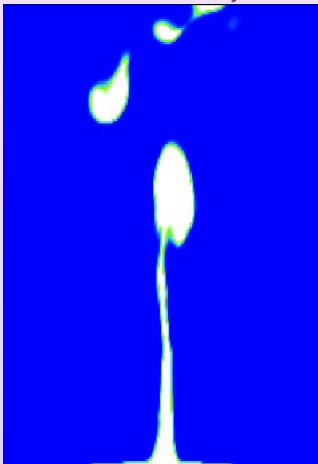
◀ Geometry

▶ Play

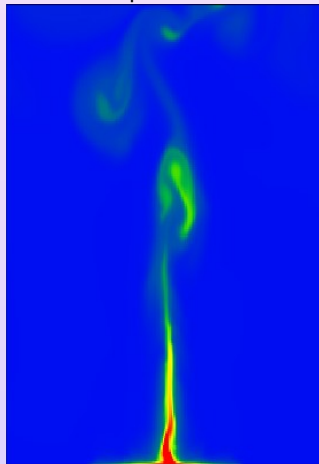
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

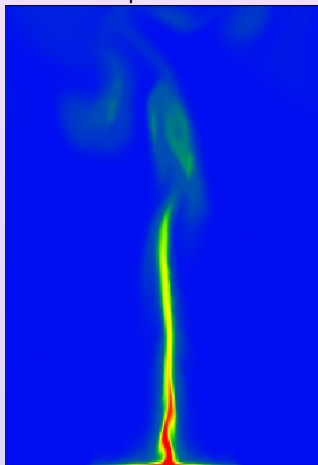
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



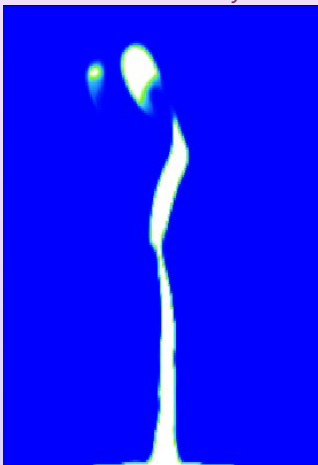
◀ Geometry

▶ Play

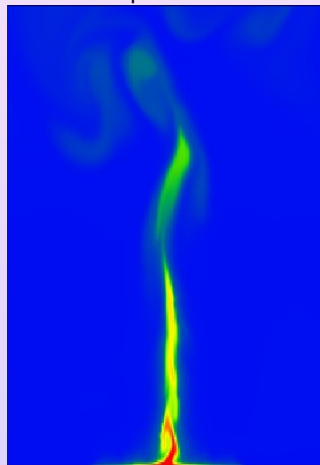
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

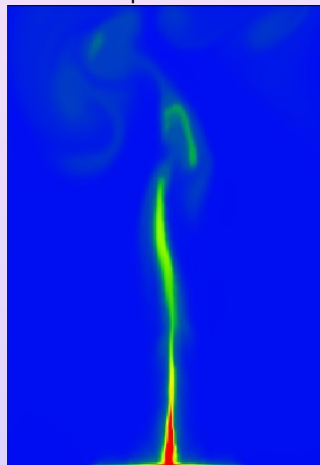
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



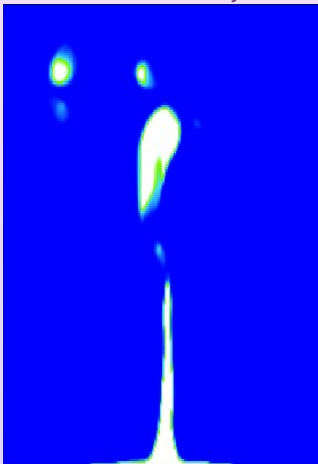
◀ Geometry

▶ Play

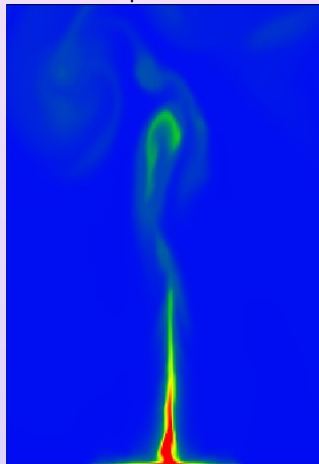
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



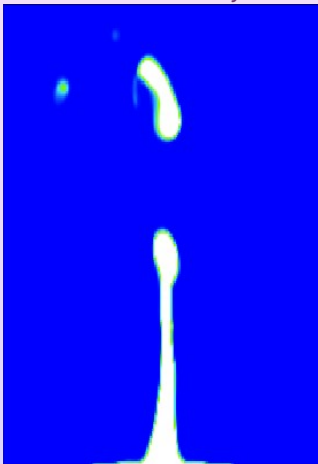
◀ Geometry

▶ Play

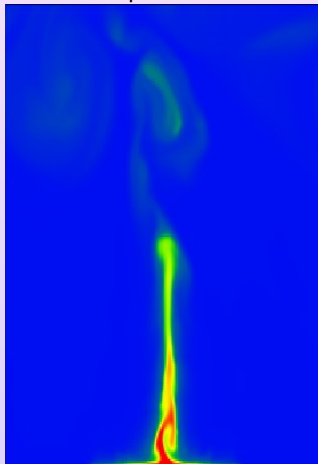
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



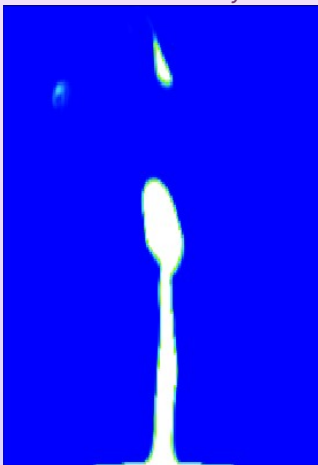
◀ Geometry

▶ Play

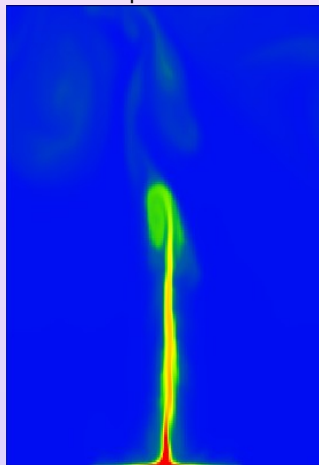
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



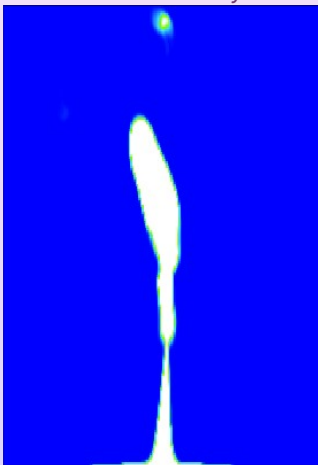
◀ Geometry

▶ Play

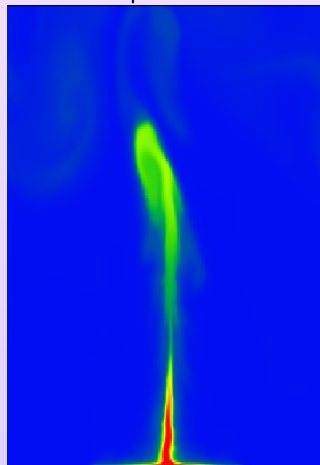
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



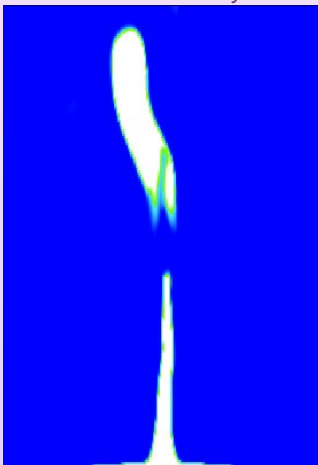
◀ Geometry

▶ Play

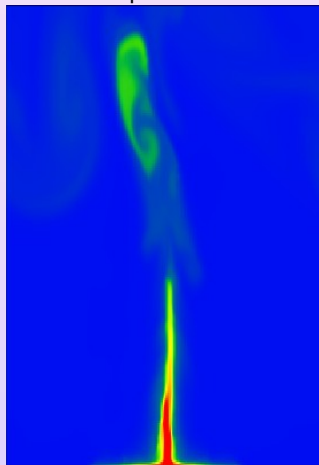
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



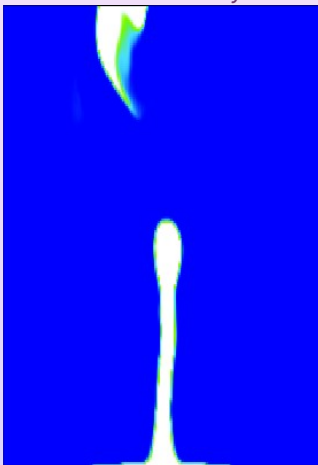
◀ Geometry

▶ Play

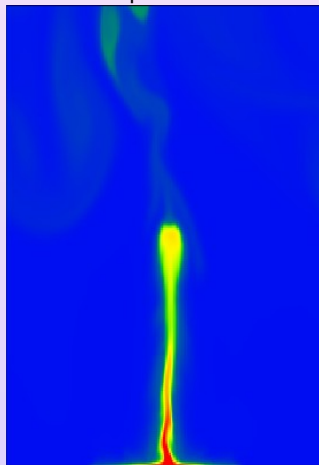
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



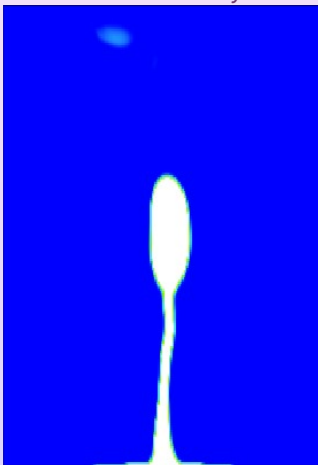
◀ Geometry

▶ Play

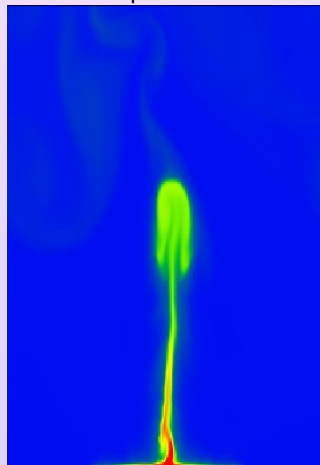
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



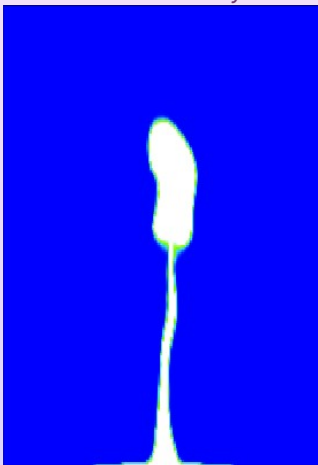
◀ Geometry

▶ Play

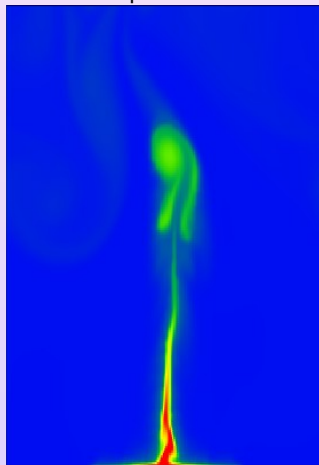
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



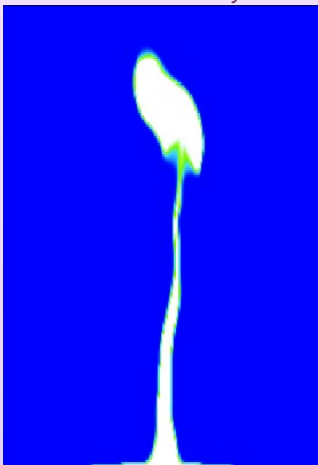
◀ Geometry

▶ Play

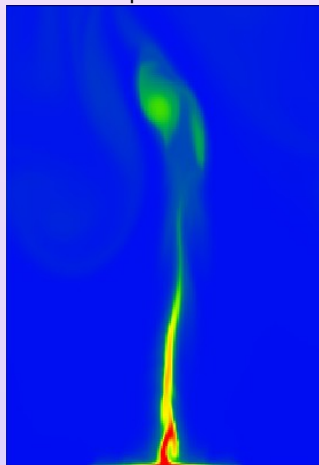
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



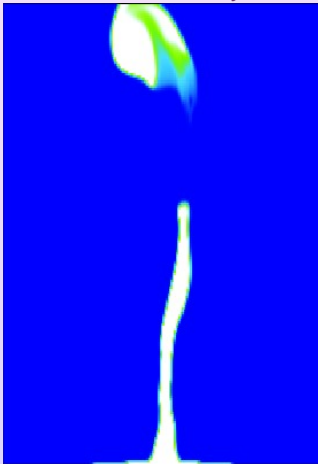
◀ Geometry

▶ Play

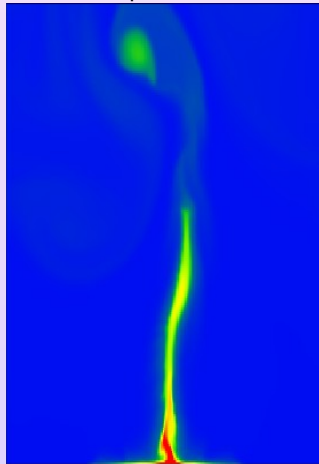
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



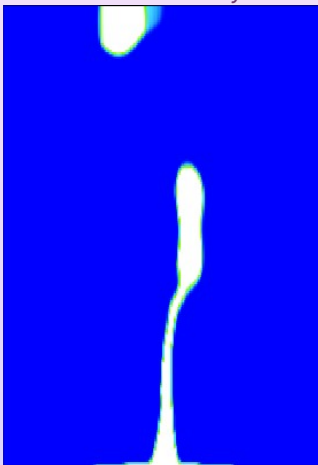
◀ Geometry

▶ Play

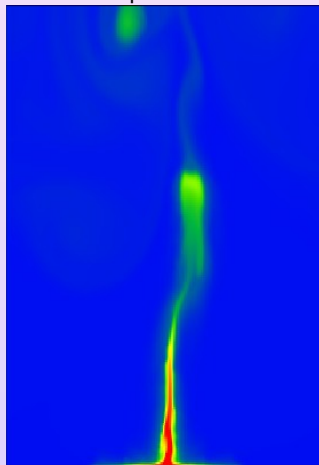
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



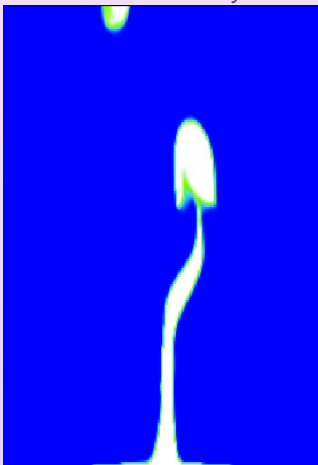
◀ Geometry

▶ Play

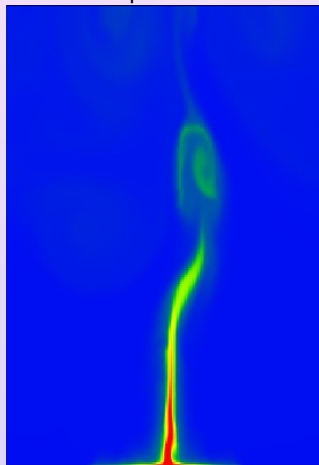
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



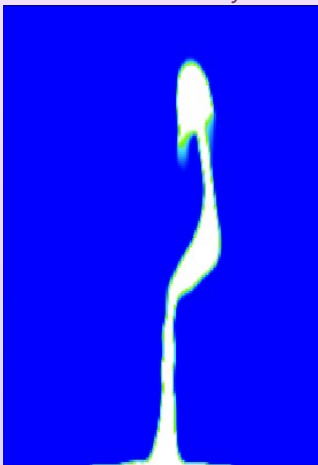
◀ Geometry

▶ Play

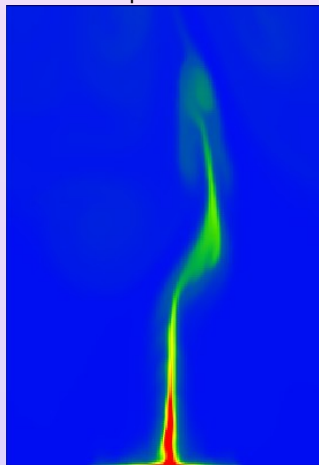
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



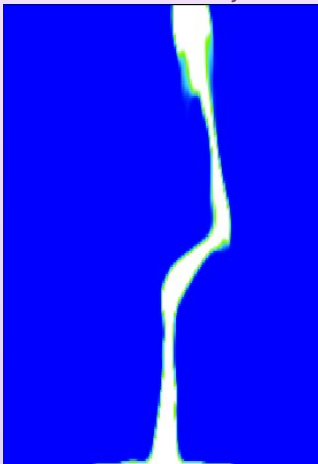
◀ Geometry

▶ Play

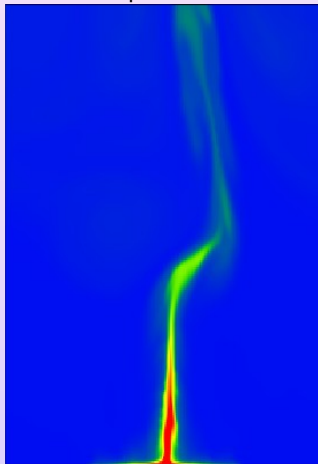
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



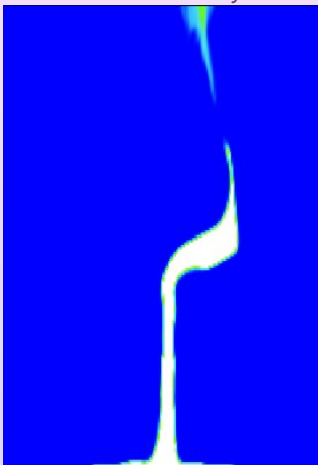
◀ Geometry

▶ Play

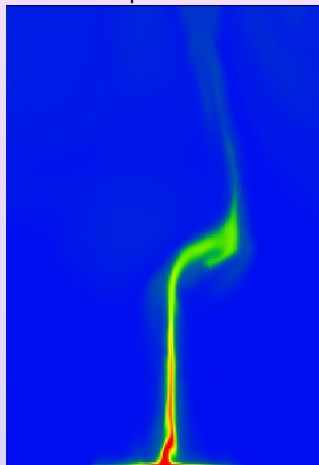
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



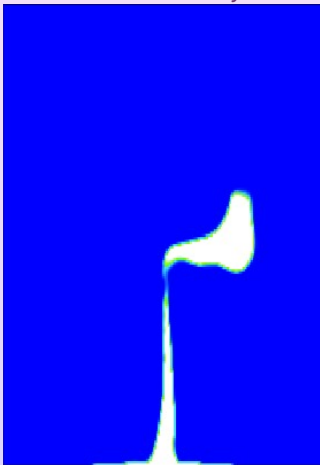
◀ Geometry

▶ Play

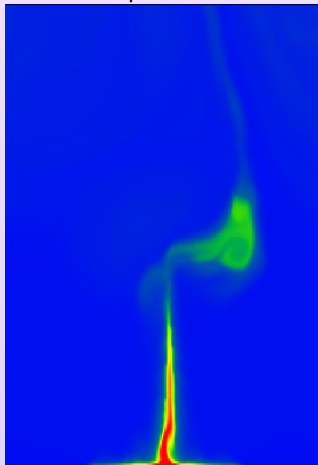
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



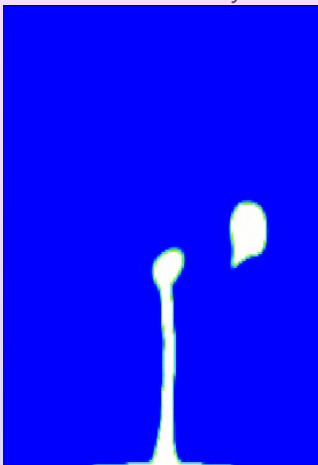
◀ Geometry

▶ Play

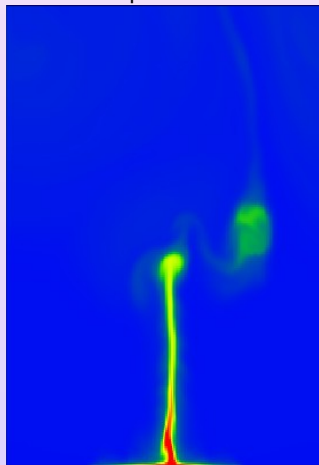
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



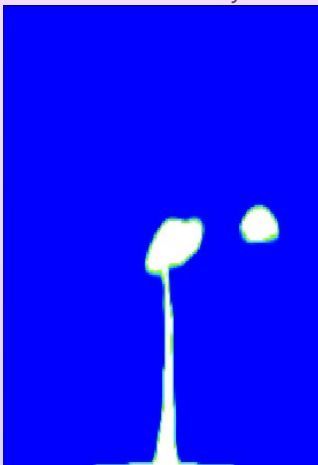
◀ Geometry

▶ Play

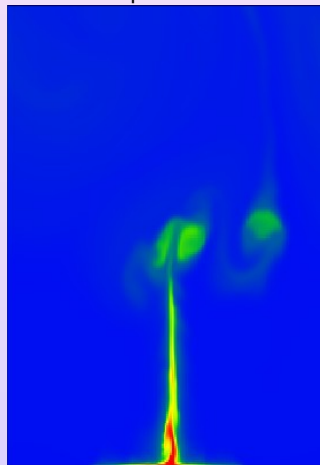
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



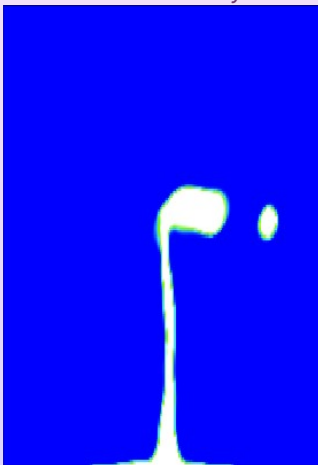
◀ Geometry

▶ Play

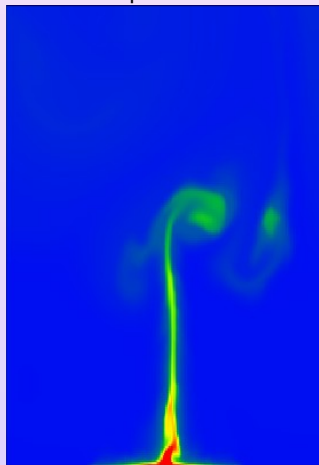
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



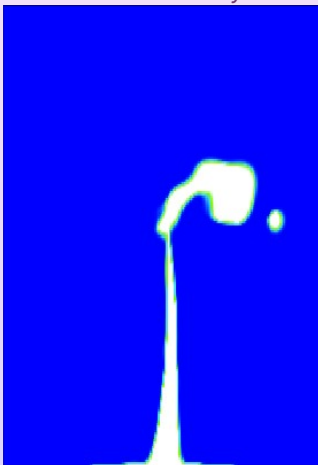
◀ Geometry

▶ Play

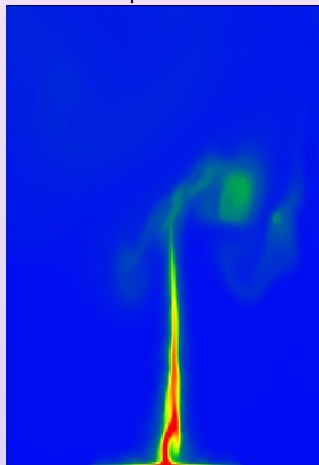
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



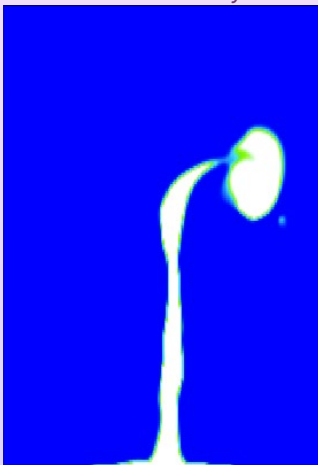
◀ Geometry

▶ Play

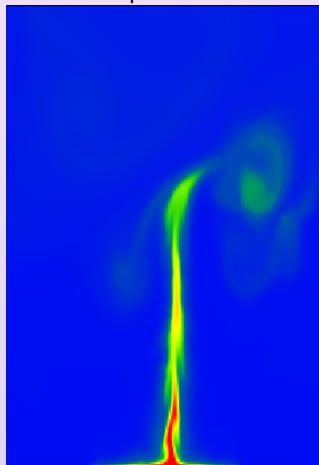
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



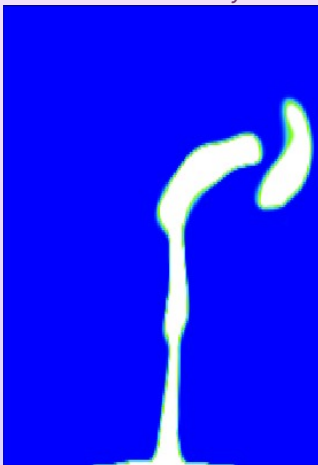
◀ Geometry

▶ Play

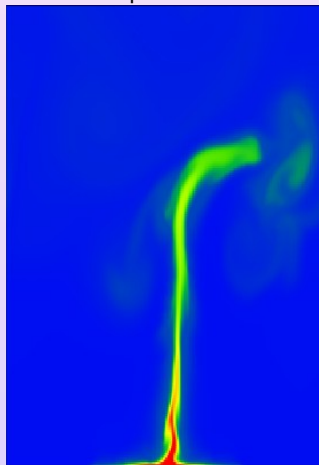
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



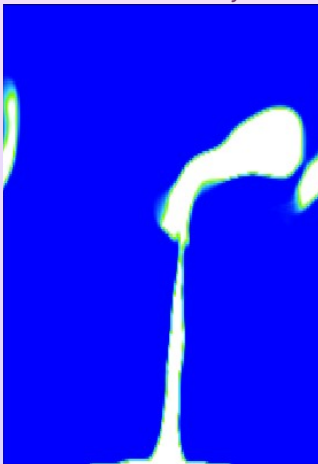
◀ Geometry

▶ Play

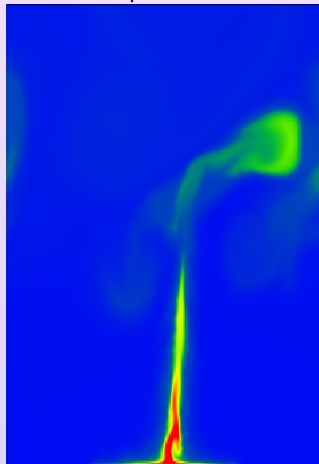
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



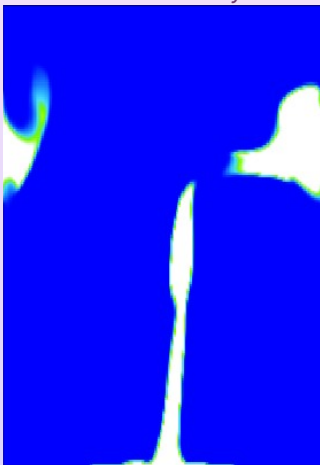
◀ Geometry

▶ Play

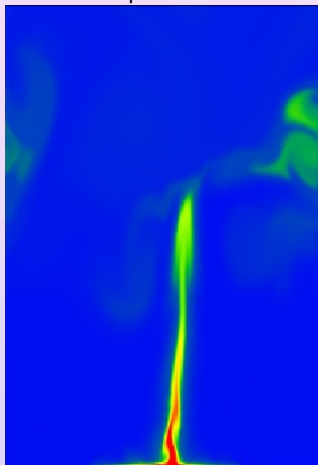
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



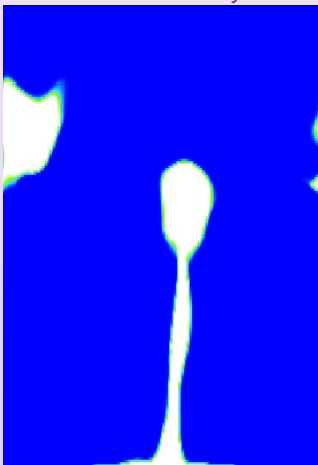
◀ Geometry

▶ Play

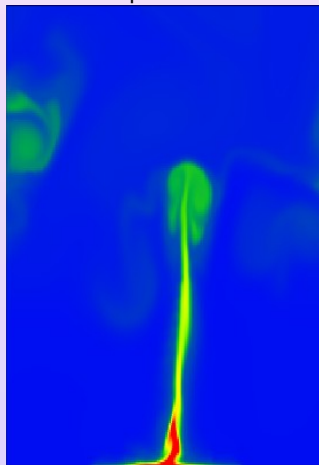
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



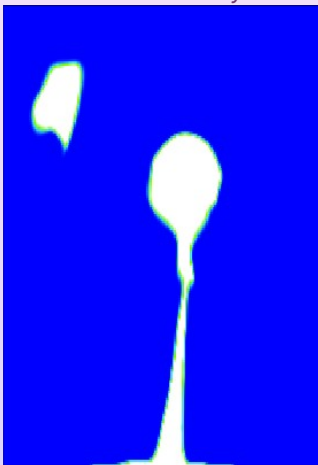
◀ Geometry

▶ Play

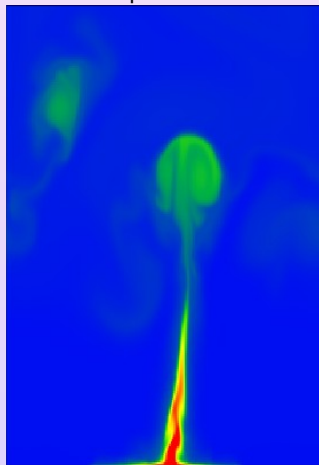
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



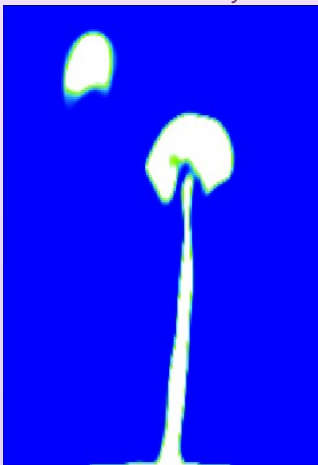
◀ Geometry

▶ Play

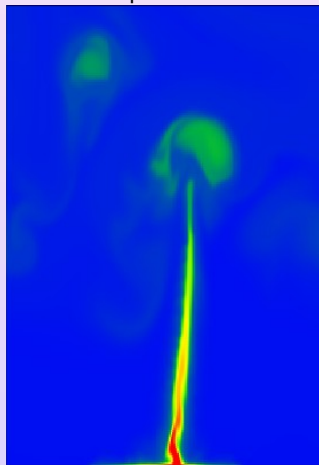
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



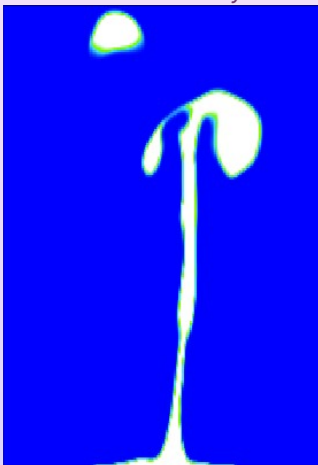
◀ Geometry

▶ Play

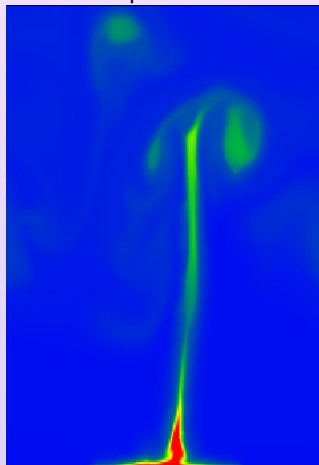
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

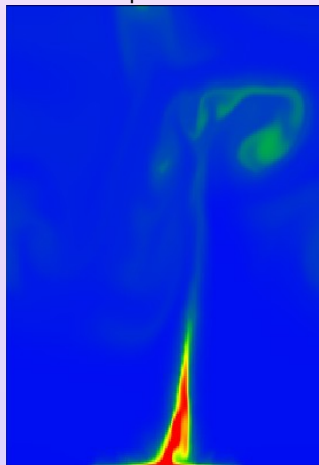
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

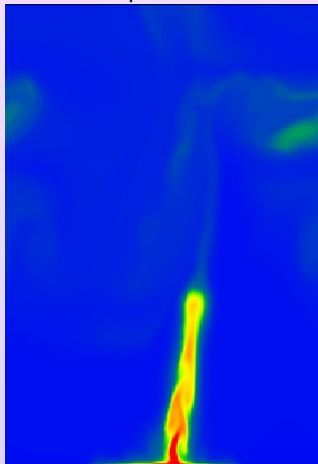
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



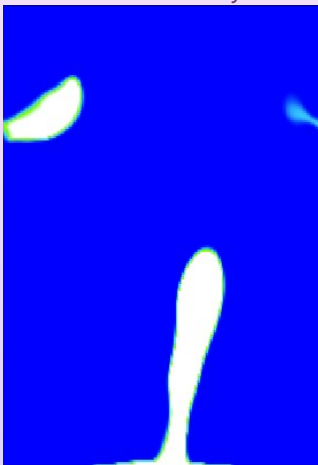
◀ Geometry

▶ Play

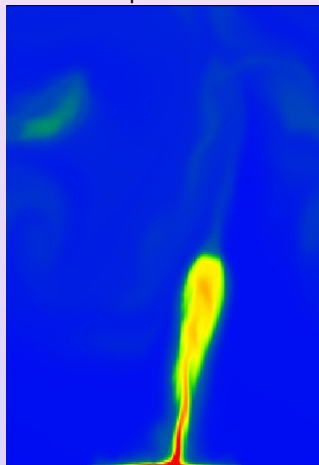
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



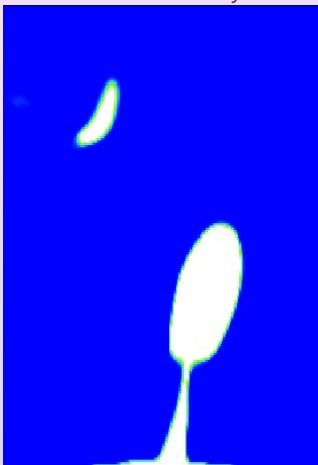
◀ Geometry

▶ Play

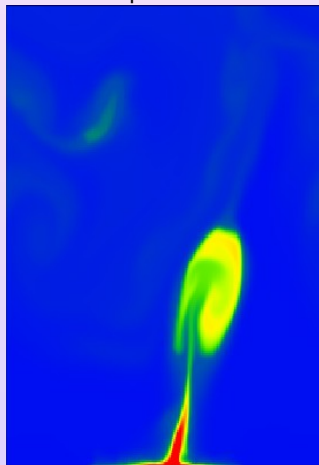
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

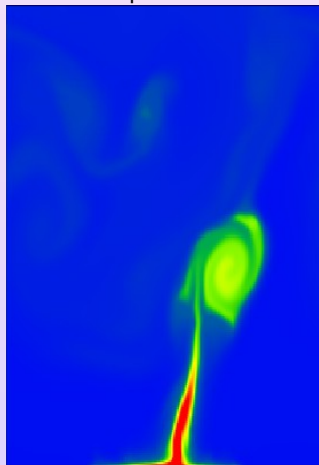
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

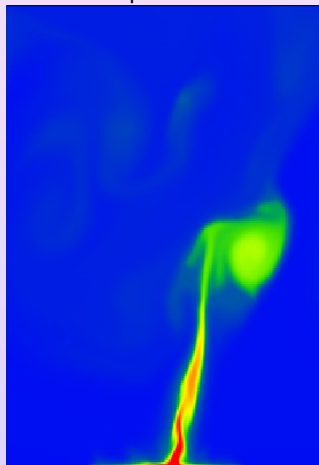
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



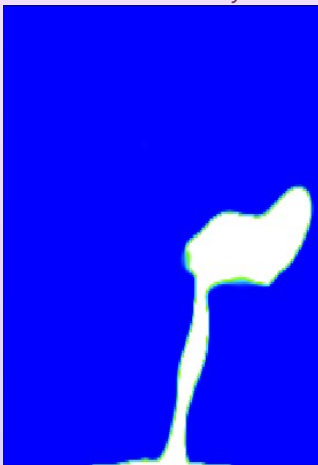
◀ Geometry

▶ Play

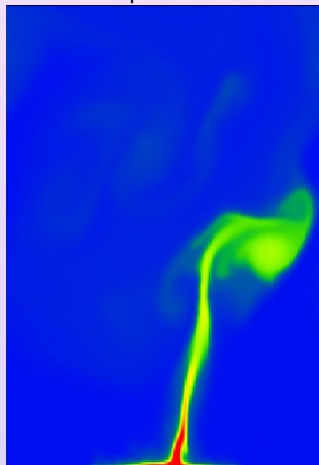
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



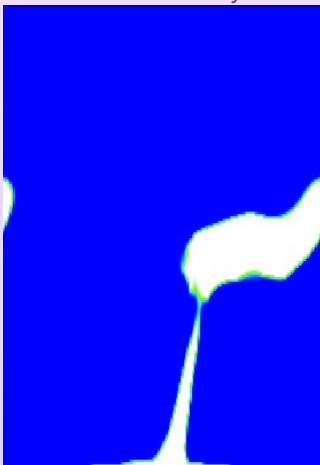
◀ Geometry

▶ Play

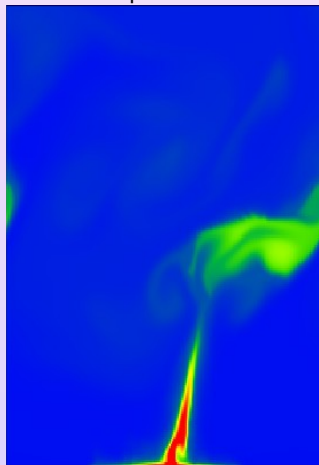
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

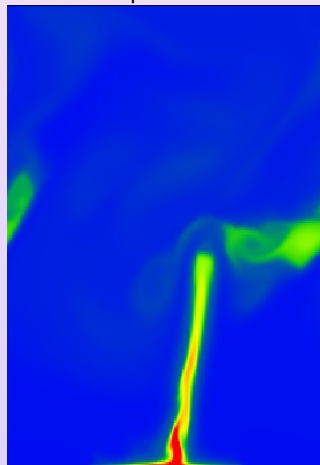
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



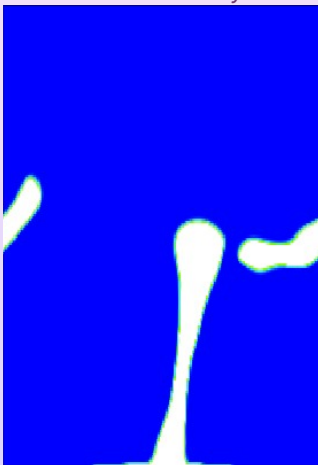
◀ Geometry

▶ Play

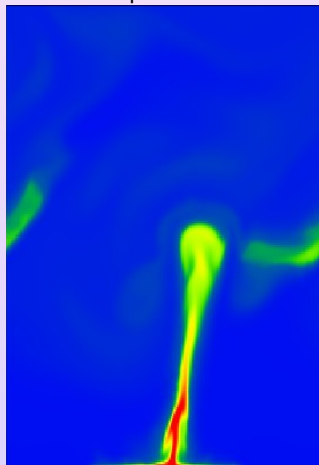
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



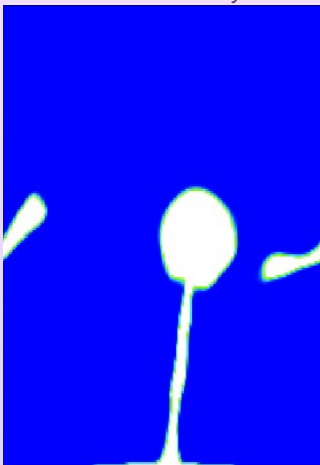
◀ Geometry

▶ Play

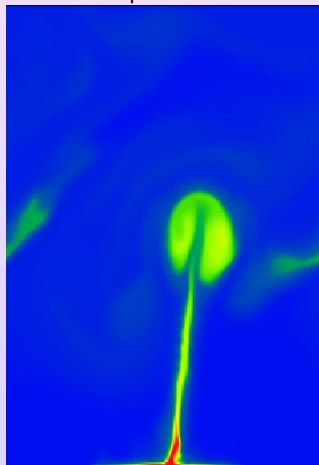
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

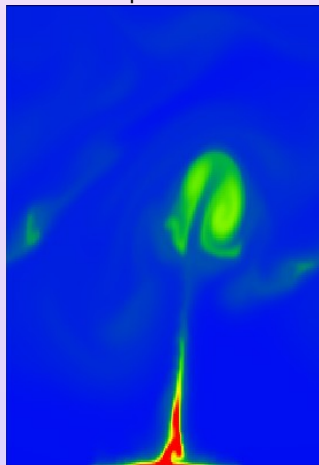
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



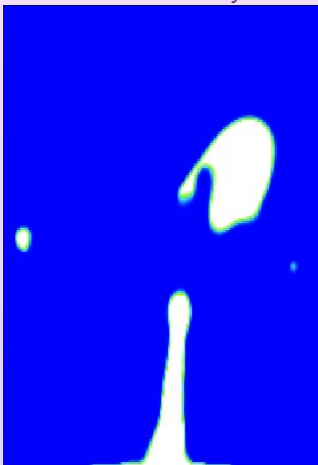
◀ Geometry

▶ Play

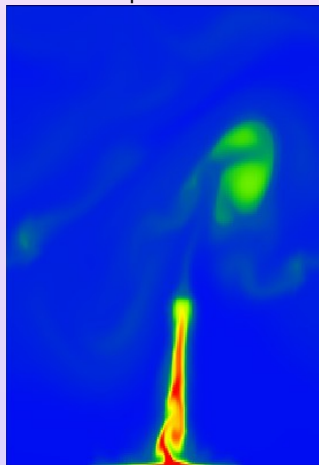
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



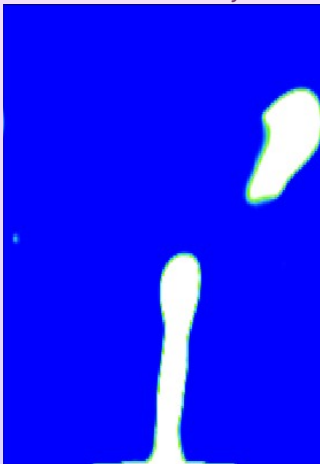
◀ Geometry

▶ Play

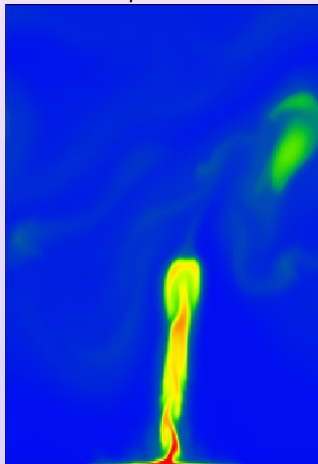
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



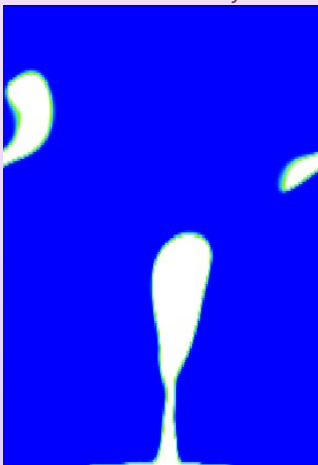
◀ Geometry

▶ Play

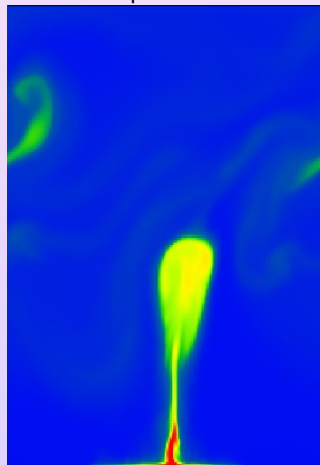
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



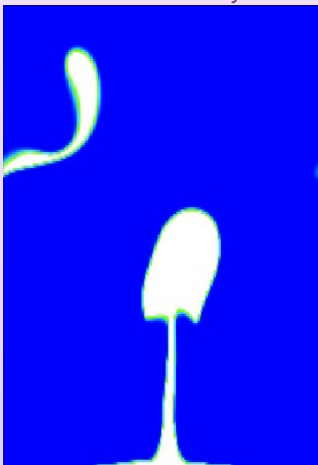
◀ Geometry

▶ Play

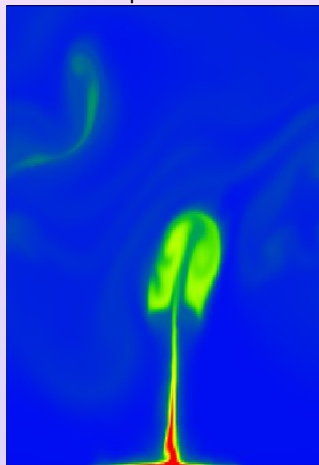
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



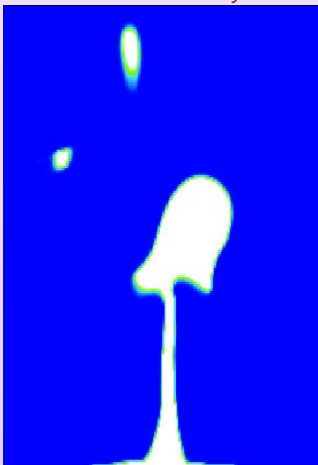
◀ Geometry

▶ Play

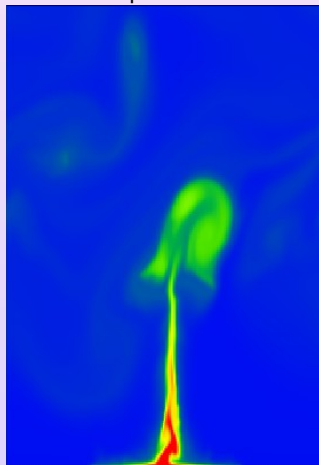
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



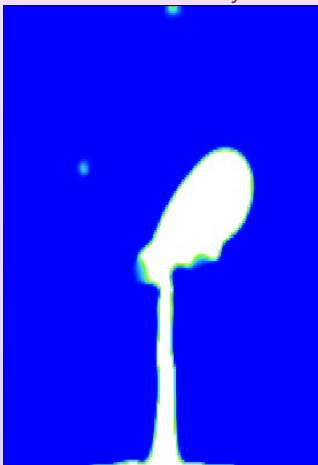
◀ Geometry

▶ Play

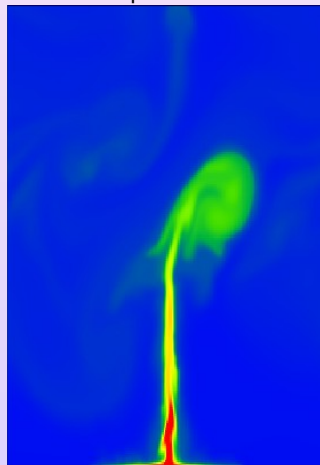
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



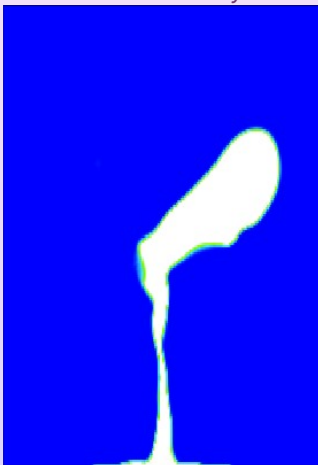
◀ Geometry

▶ Play

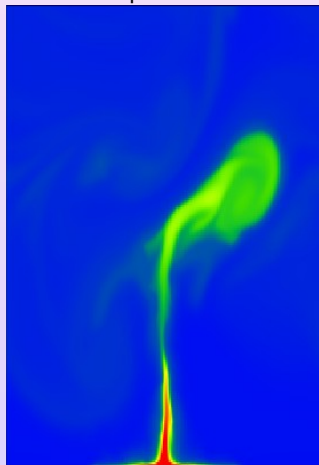
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



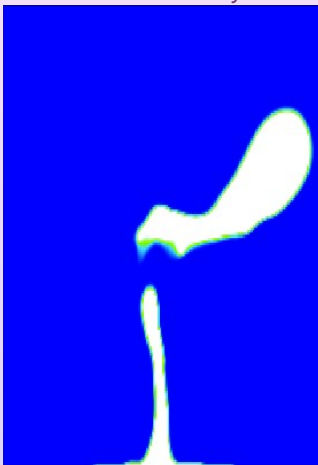
◀ Geometry

▶ Play

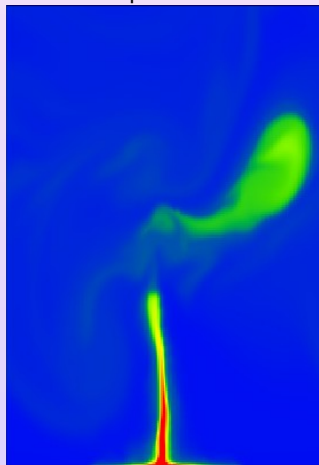
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T



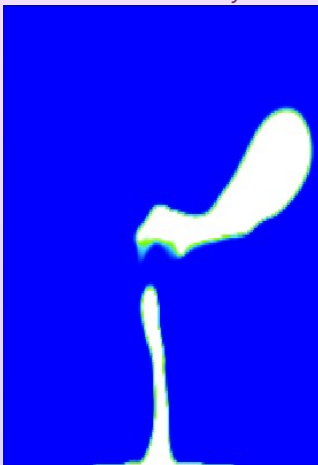
◀ Geometry

▶ Play

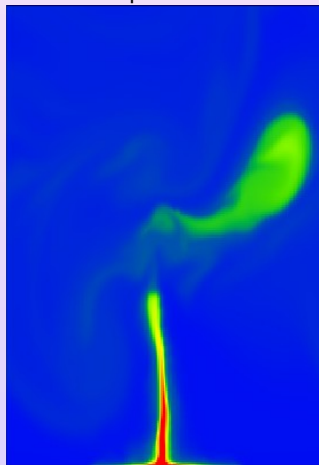
▶▶ Skip

Nucleating Bubble

Mass Fraction y



Temperature T

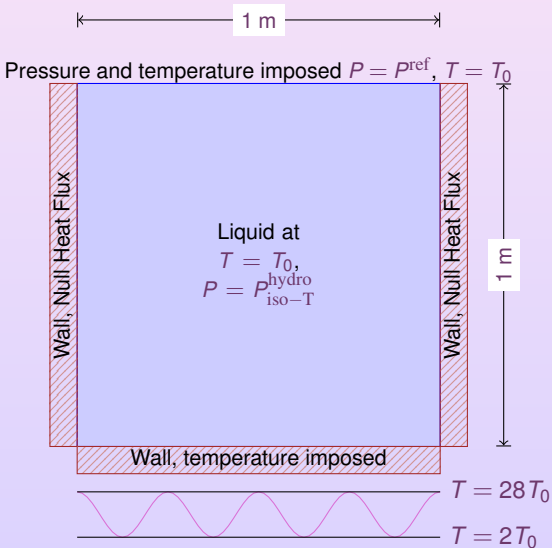


◀ Geometry

▶ Play

▶▶ Skip

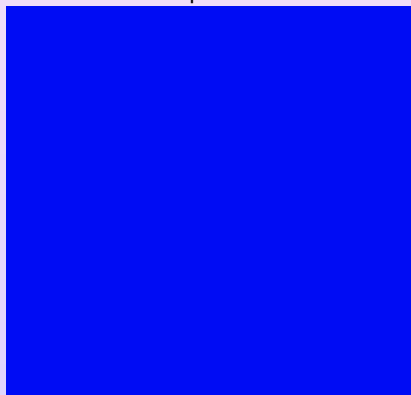
Film



Film

Mass Fraction y

Temperature T



◀ Geometry

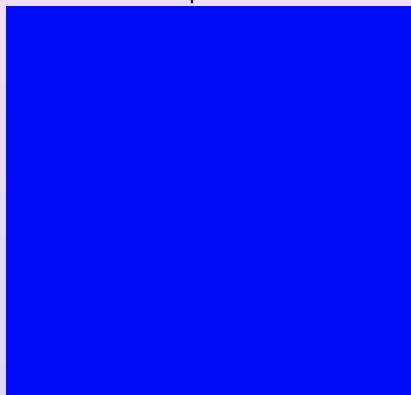
▶ Play

▶▶ Skip

Film

Mass Fraction y

Temperature T



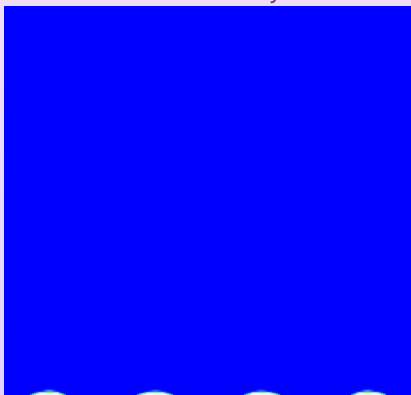
◀ Geometry

▶ Play

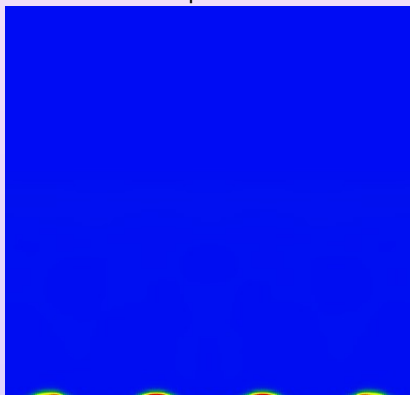
▶▶ Skip

Film

Mass Fraction y



Temperature T



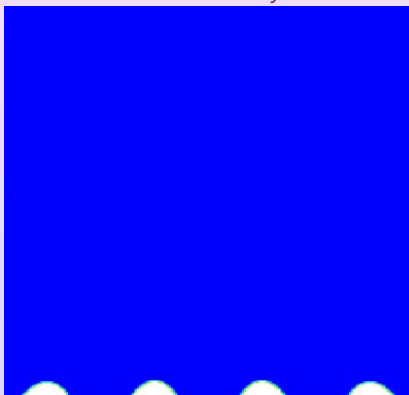
◀ Geometry

▶ Play

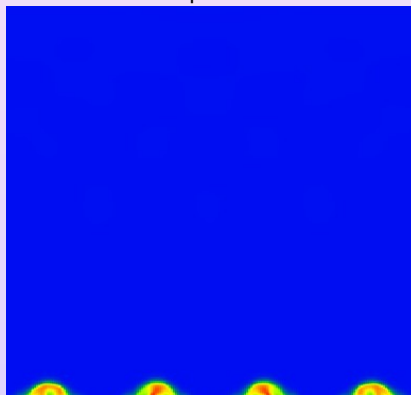
▶▶ Skip

Film

Mass Fraction y



Temperature T



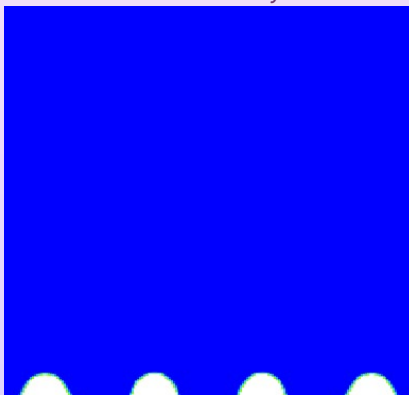
◀ Geometry

▶ Play

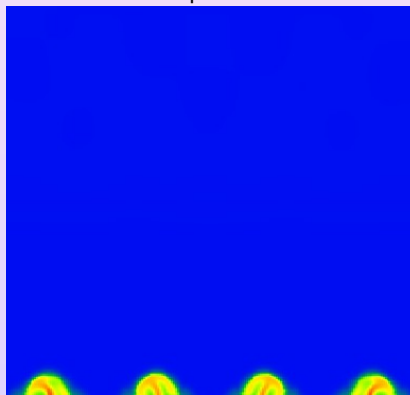
▶▶ Skip

Film

Mass Fraction y



Temperature T



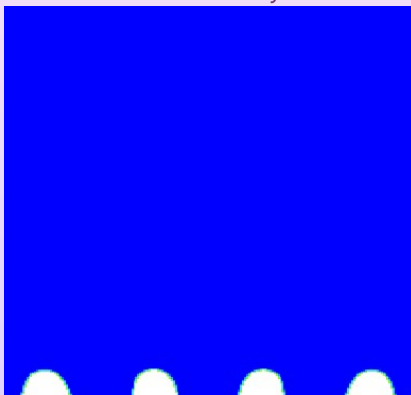
◀ Geometry

▶ Play

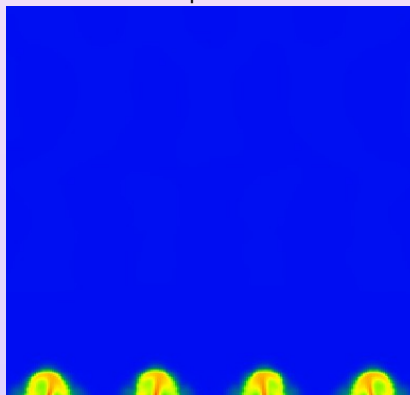
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

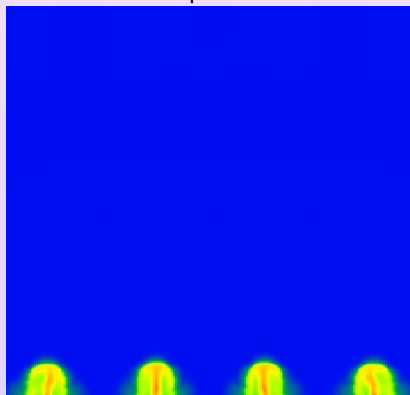
▶▶ Skip

Film

Mass Fraction y



Temperature T



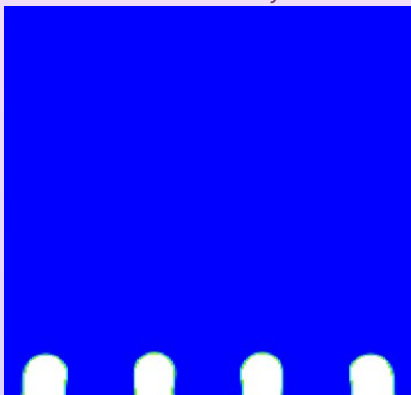
◀ Geometry

▶ Play

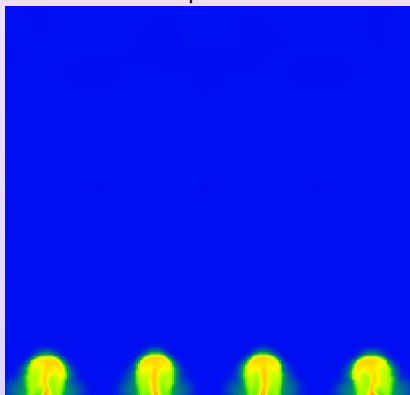
▶▶ Skip

Film

Mass Fraction y



Temperature T



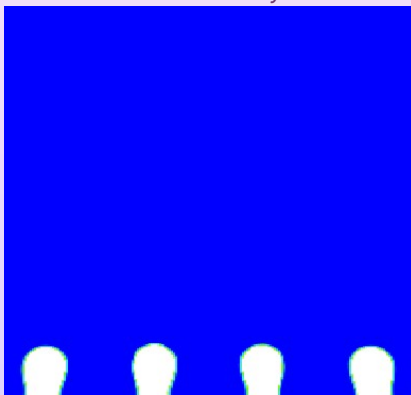
◀ Geometry

▶ Play

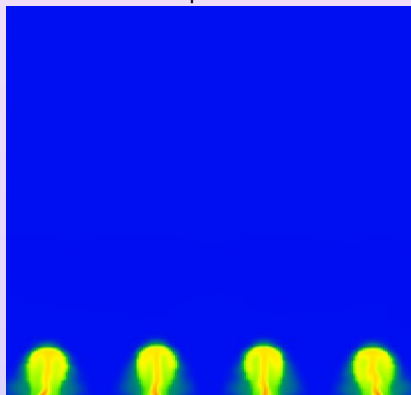
▶▶ Skip

Film

Mass Fraction y



Temperature T



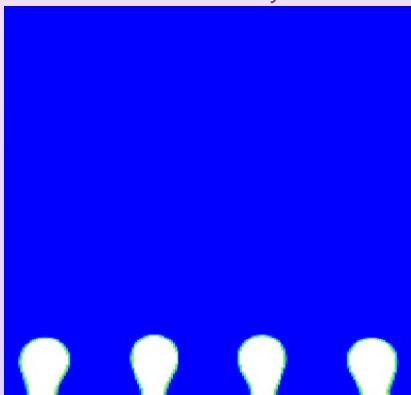
◀ Geometry

▶ Play

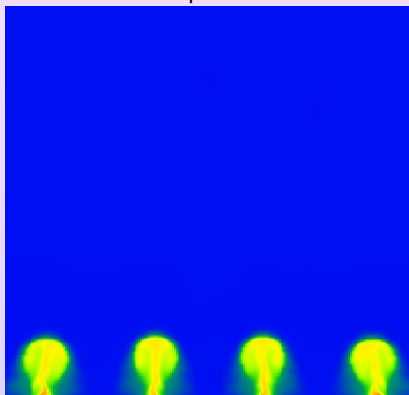
▶▶ Skip

Film

Mass Fraction y



Temperature T



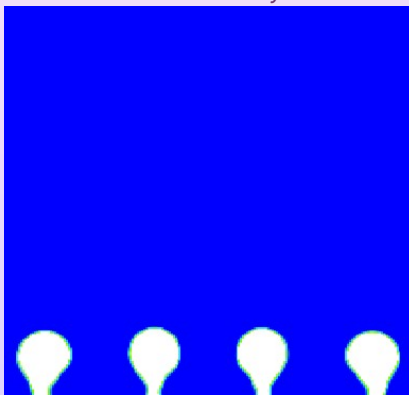
◀ Geometry

▶ Play

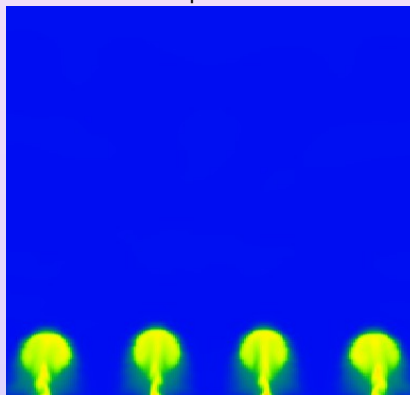
▶▶ Skip

Film

Mass Fraction y



Temperature T



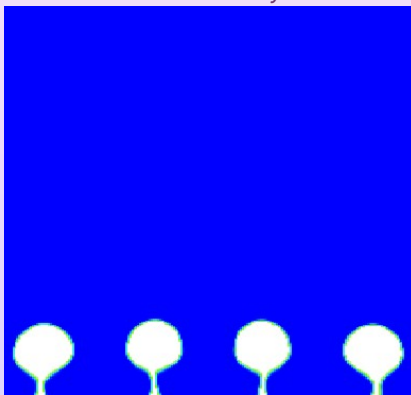
◀ Geometry

▶ Play

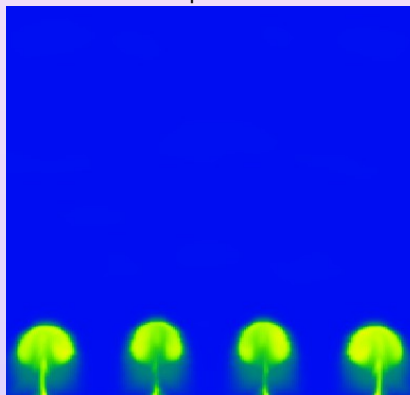
▶▶ Skip

Film

Mass Fraction y



Temperature T



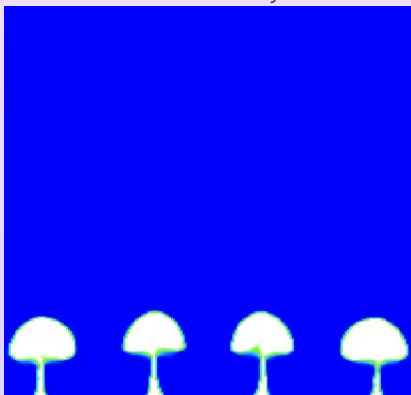
◀ Geometry

▶ Play

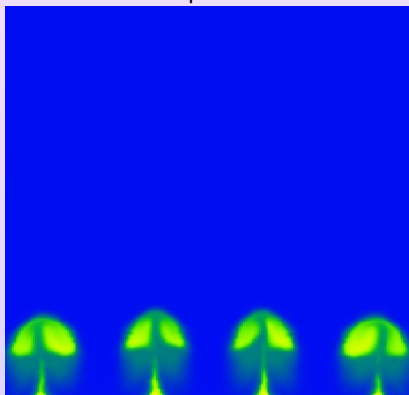
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

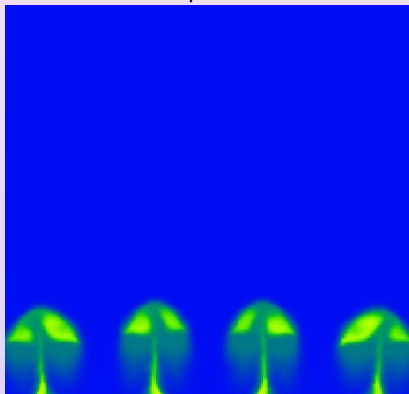
▶▶ Skip

Film

Mass Fraction y



Temperature T



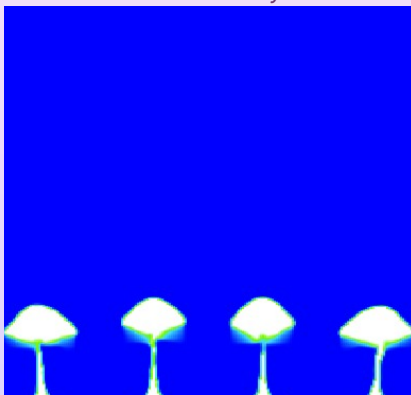
◀ Geometry

▶ Play

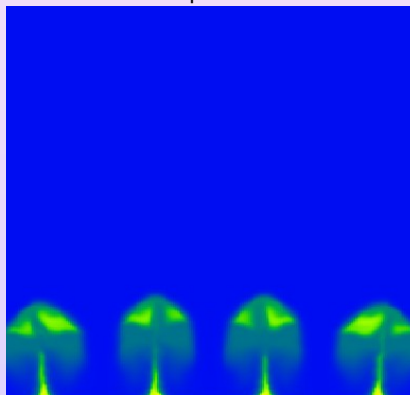
▶▶ Skip

Film

Mass Fraction y



Temperature T



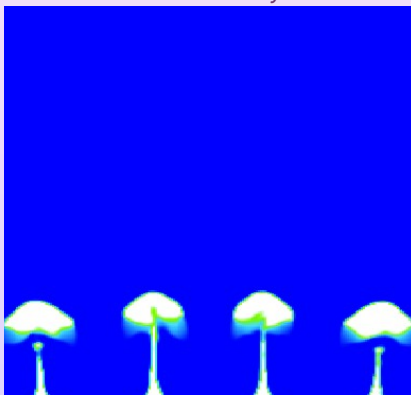
◀ Geometry

▶ Play

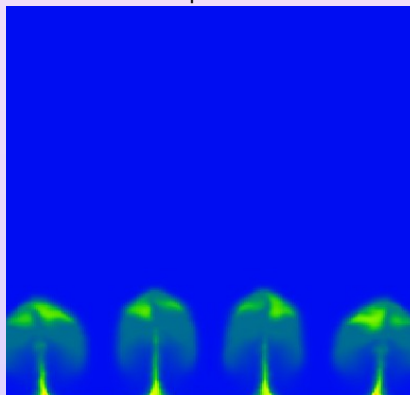
▶▶ Skip

Film

Mass Fraction y



Temperature T



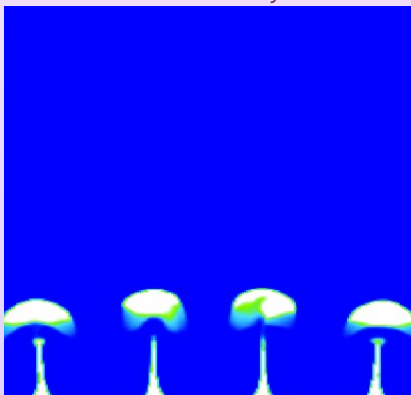
◀ Geometry

▶ Play

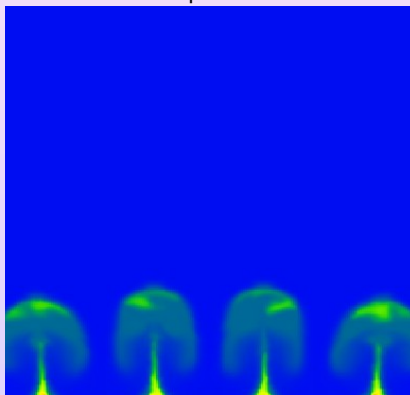
▶▶ Skip

Film

Mass Fraction y



Temperature T



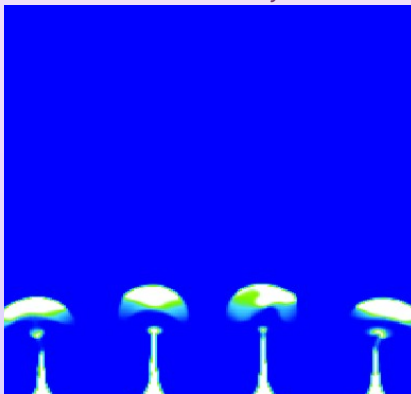
◀ Geometry

▶ Play

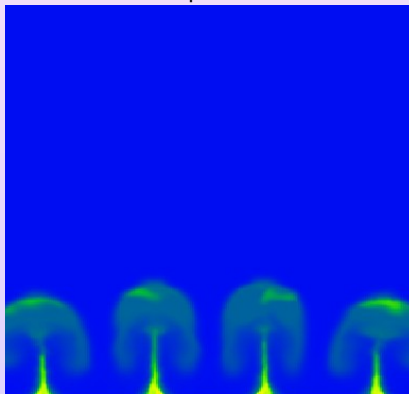
▶▶ Skip

Film

Mass Fraction y



Temperature T



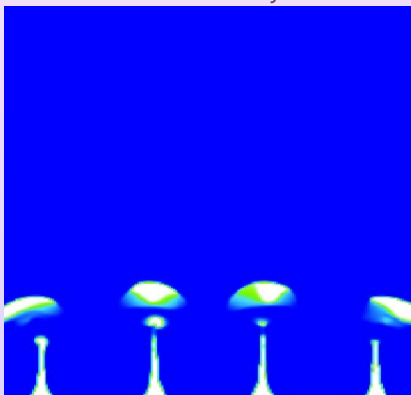
◀ Geometry

▶ Play

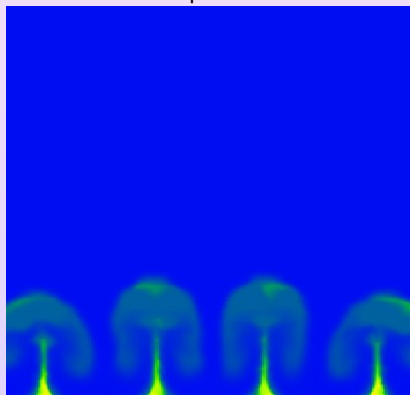
▶▶ Skip

Film

Mass Fraction y



Temperature T



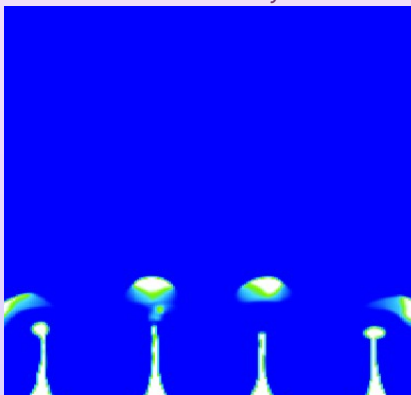
◀ Geometry

▶ Play

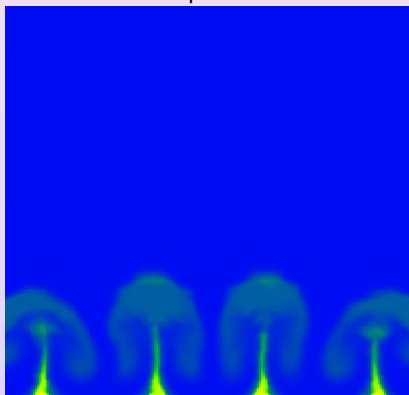
▶▶ Skip

Film

Mass Fraction y



Temperature T



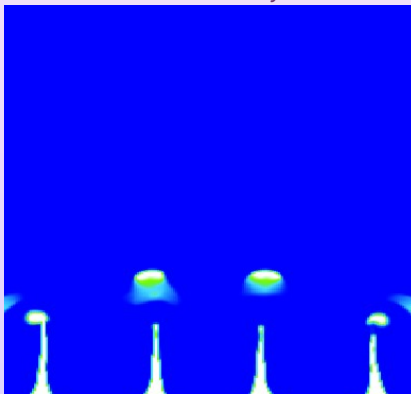
◀ Geometry

▶ Play

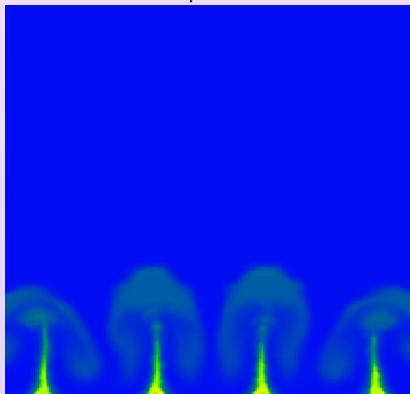
▶▶ Skip

Film

Mass Fraction y



Temperature T



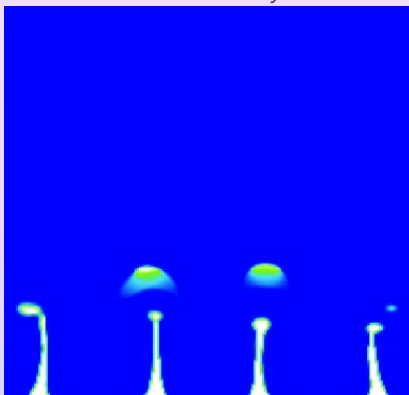
◀ Geometry

▶ Play

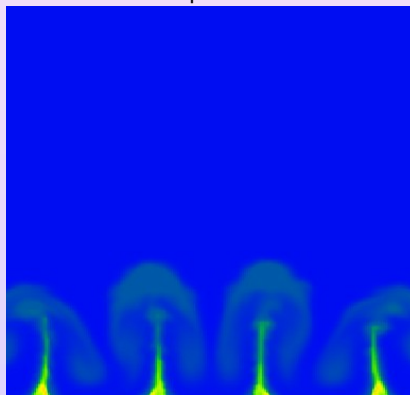
▶▶ Skip

Film

Mass Fraction y



Temperature T



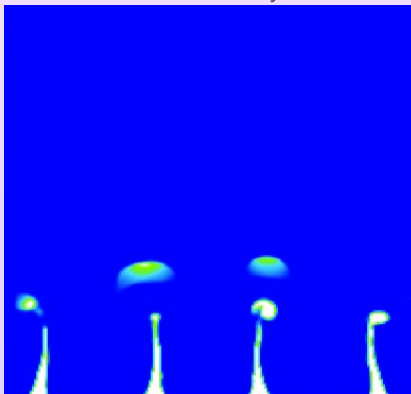
◀ Geometry

▶ Play

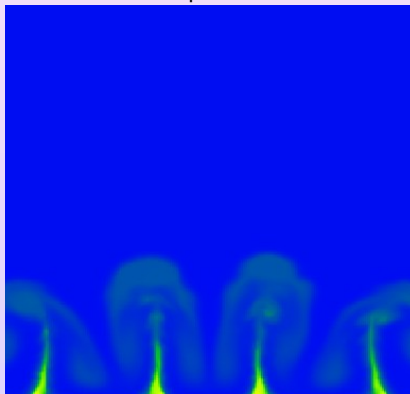
▶▶ Skip

Film

Mass Fraction y



Temperature T



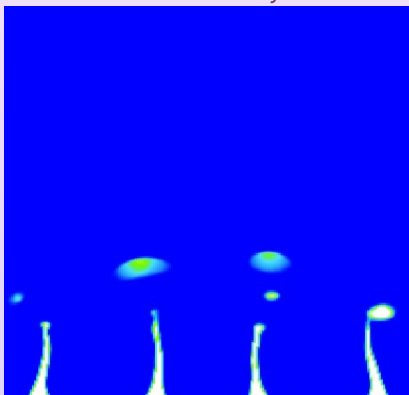
◀ Geometry

▶ Play

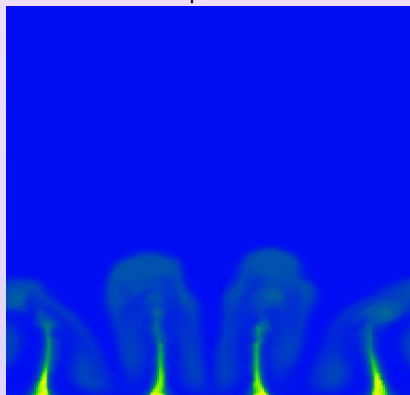
▶▶ Skip

Film

Mass Fraction y



Temperature T



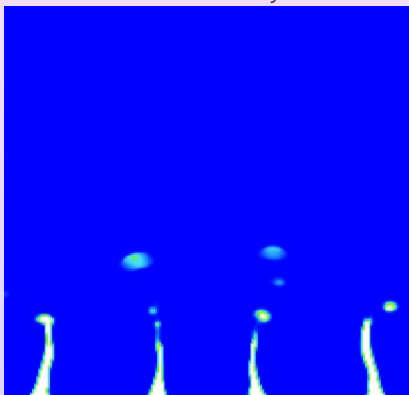
◀ Geometry

▶ Play

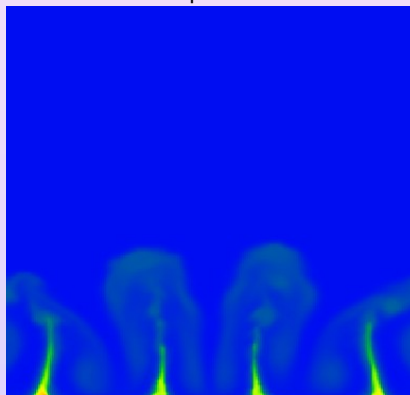
▶▶ Skip

Film

Mass Fraction y



Temperature T



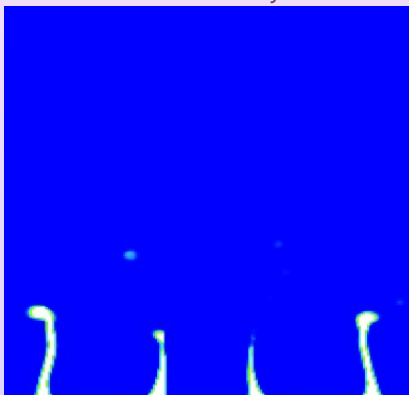
◀ Geometry

▶ Play

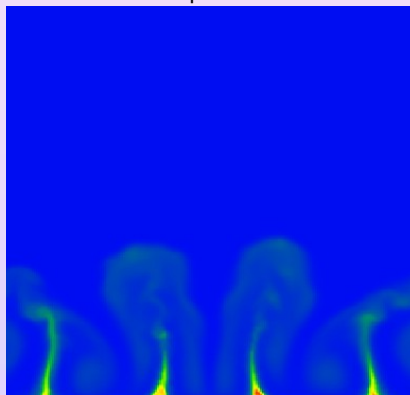
▶▶ Skip

Film

Mass Fraction y



Temperature T



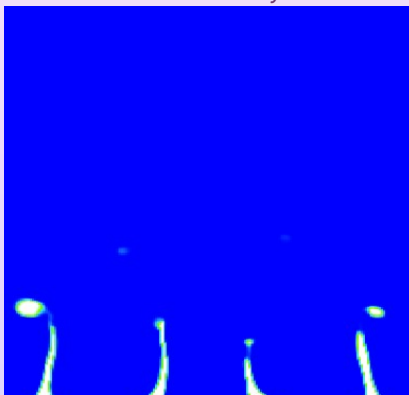
◀ Geometry

▶ Play

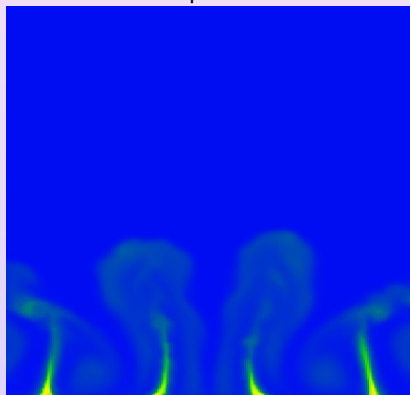
▶▶ Skip

Film

Mass Fraction y



Temperature T



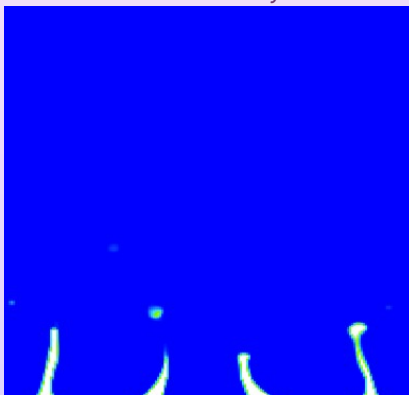
◀ Geometry

▶ Play

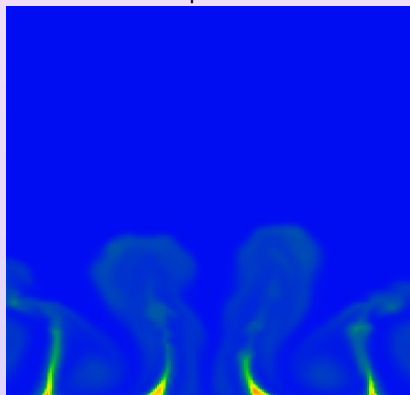
▶▶ Skip

Film

Mass Fraction y



Temperature T



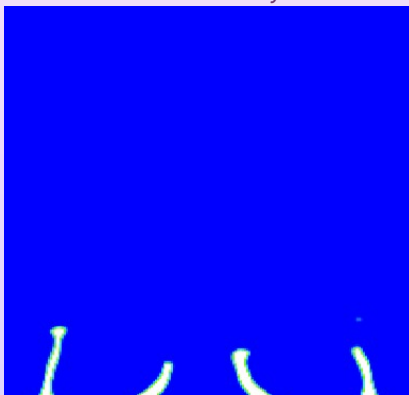
◀ Geometry

▶ Play

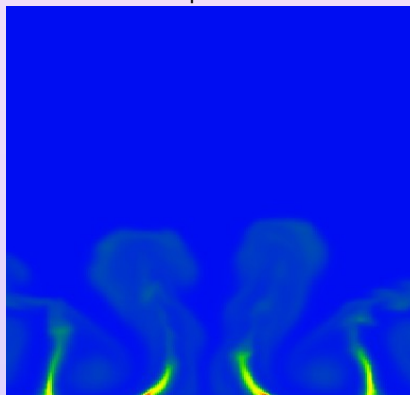
▶▶ Skip

Film

Mass Fraction y



Temperature T



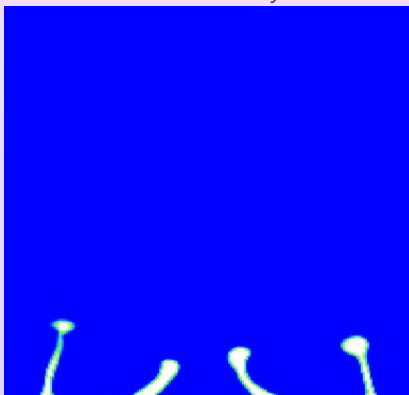
◀ Geometry

▶ Play

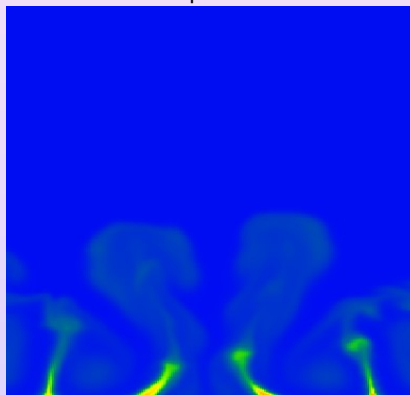
▶▶ Skip

Film

Mass Fraction y



Temperature T



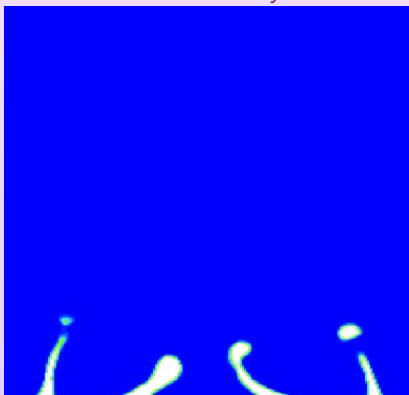
◀ Geometry

▶ Play

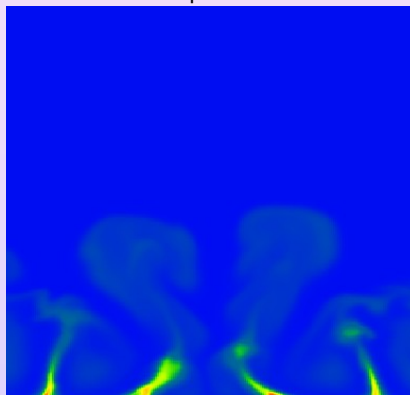
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

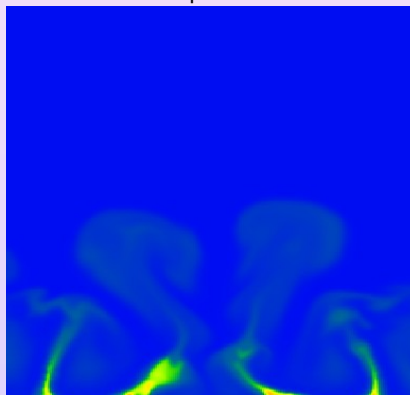
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

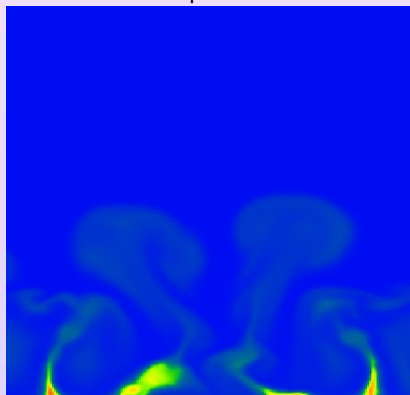
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

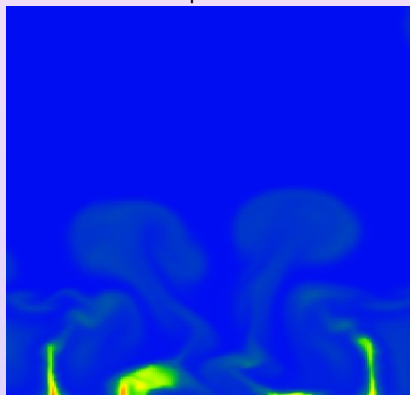
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

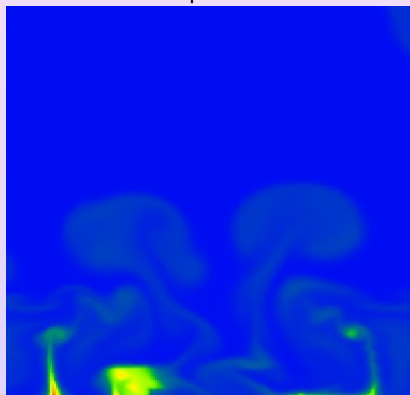
▶▶ Skip

Film

Mass Fraction y



Temperature T



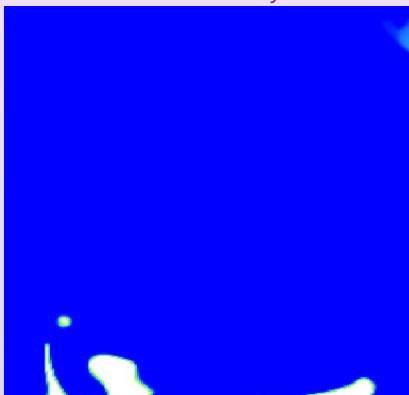
◀ Geometry

▶ Play

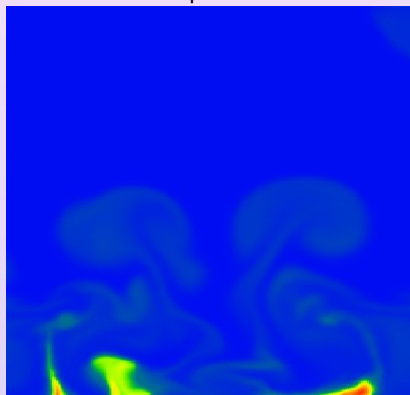
▶▶ Skip

Film

Mass Fraction y



Temperature T



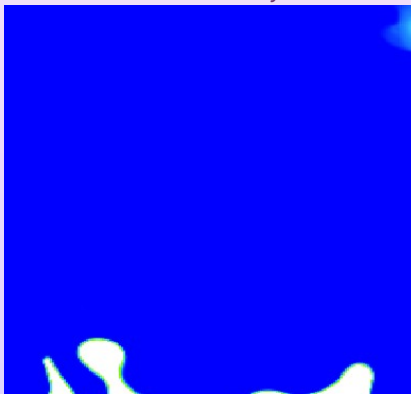
◀ Geometry

▶ Play

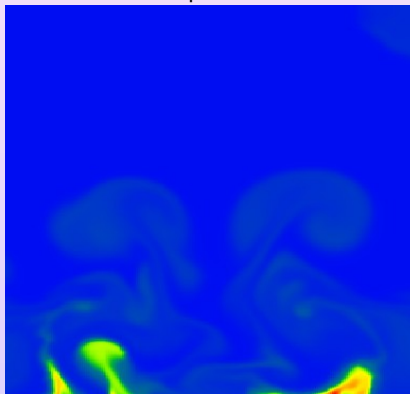
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

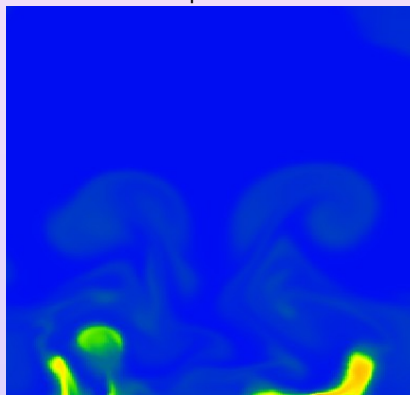
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

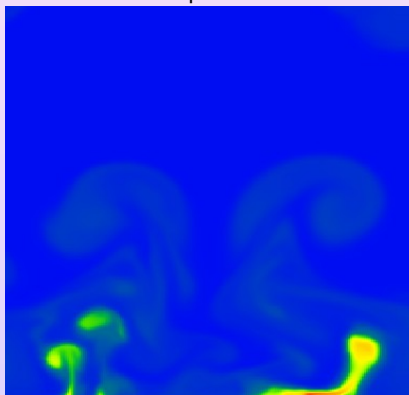
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

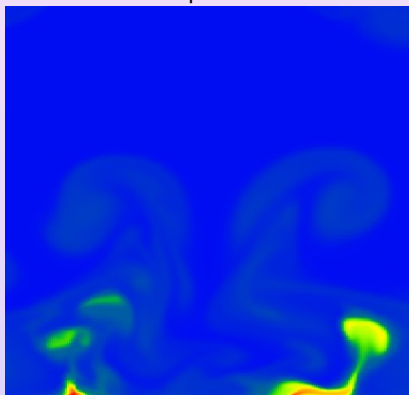
▶▶ Skip

Film

Mass Fraction y



Temperature T



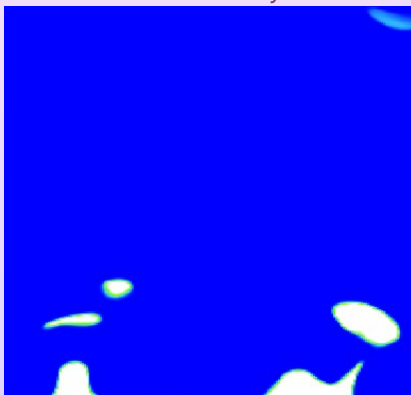
◀ Geometry

▶ Play

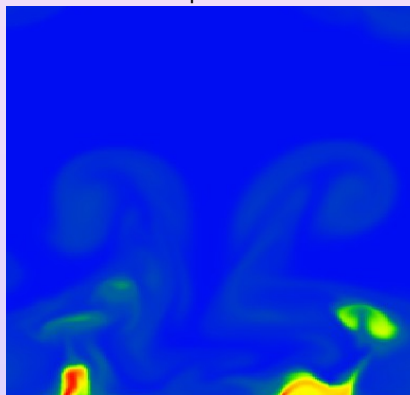
▶▶ Skip

Film

Mass Fraction y



Temperature T



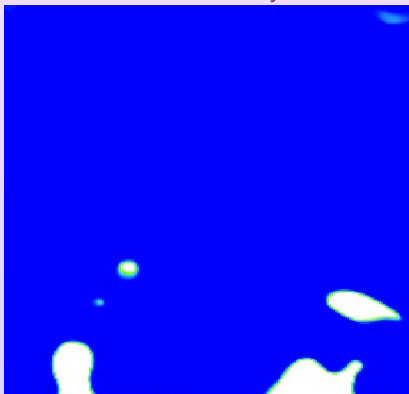
◀ Geometry

▶ Play

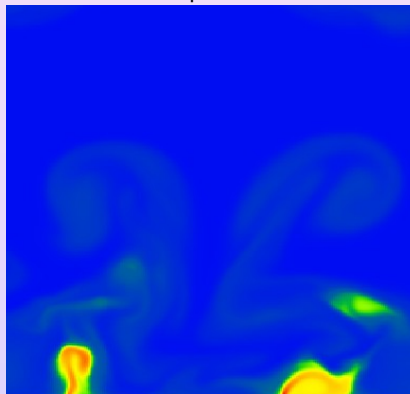
▶▶ Skip

Film

Mass Fraction y



Temperature T



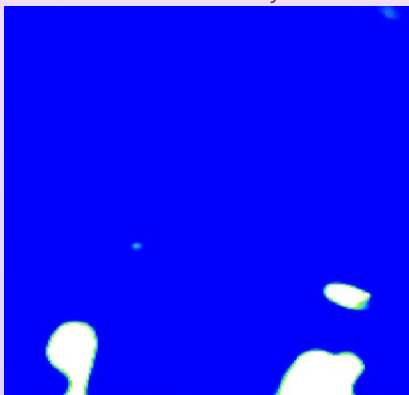
◀ Geometry

▶ Play

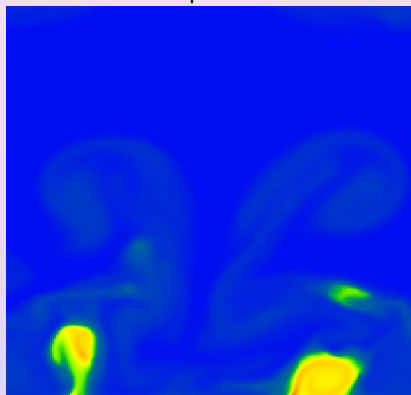
▶▶ Skip

Film

Mass Fraction y



Temperature T



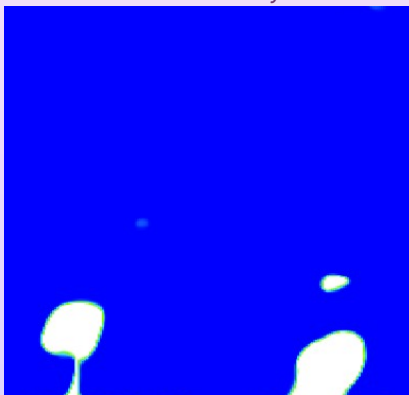
◀ Geometry

▶ Play

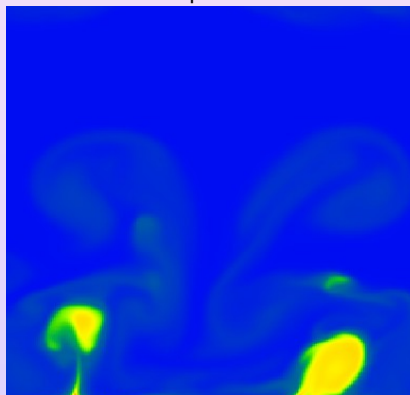
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

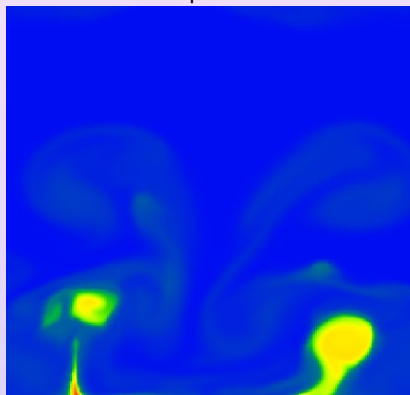
▶▶ Skip

Film

Mass Fraction y



Temperature T



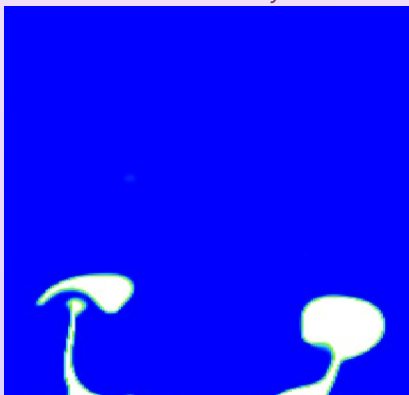
◀ Geometry

▶ Play

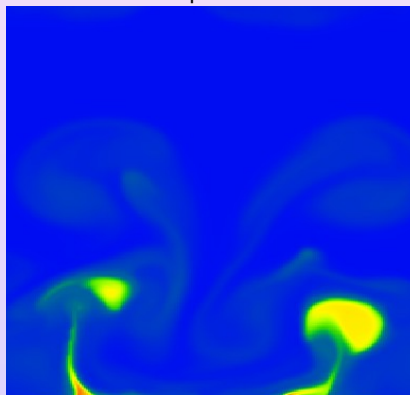
▶▶ Skip

Film

Mass Fraction y



Temperature T



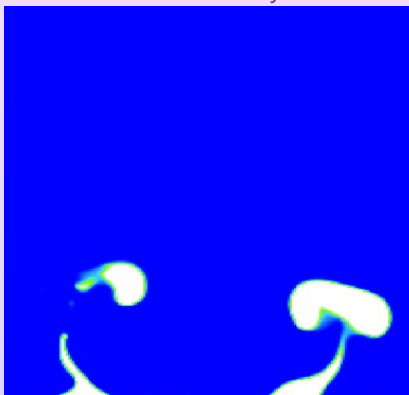
◀ Geometry

▶ Play

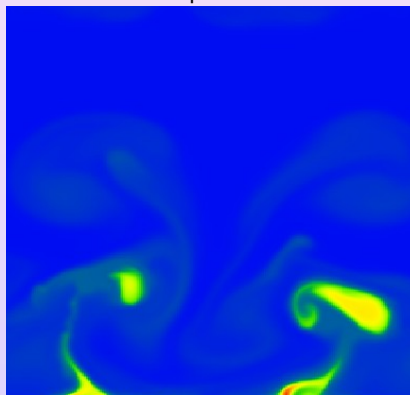
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

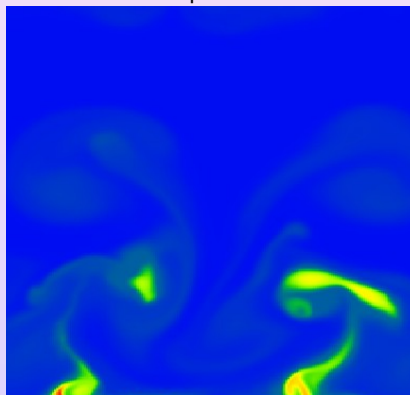
▶▶ Skip

Film

Mass Fraction y



Temperature T



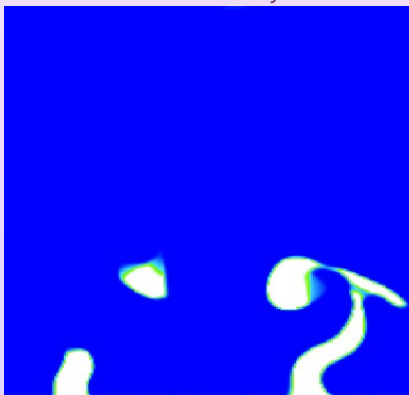
◀ Geometry

▶ Play

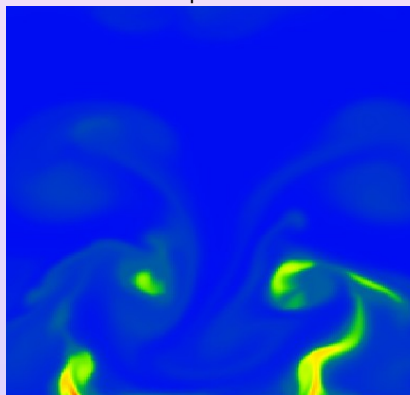
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

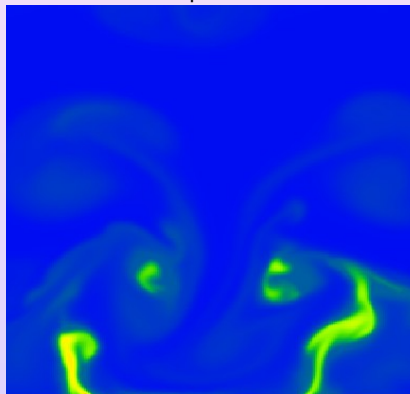
▶▶ Skip

Film

Mass Fraction y



Temperature T



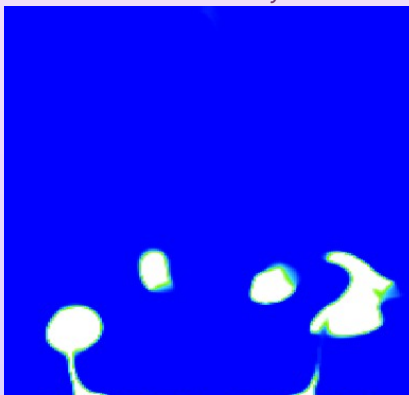
◀ Geometry

▶ Play

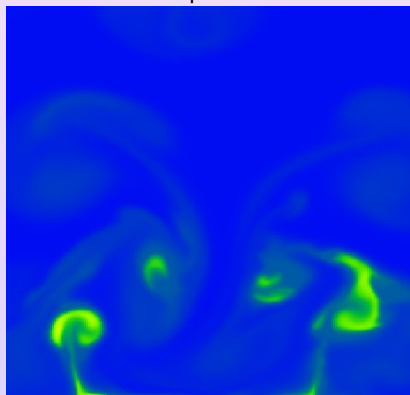
▶▶ Skip

Film

Mass Fraction y



Temperature T



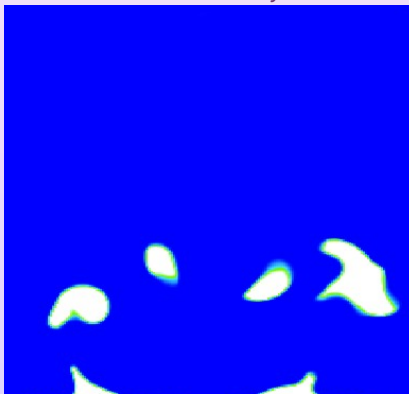
◀ Geometry

▶ Play

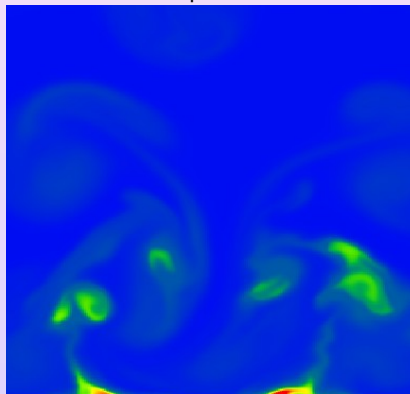
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

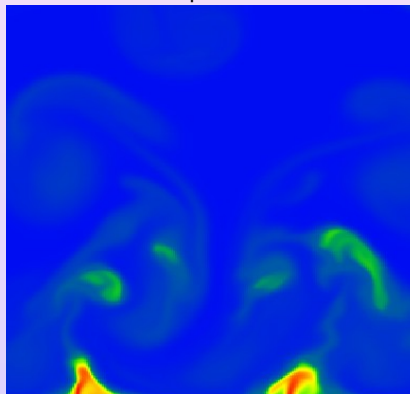
▶▶ Skip

Film

Mass Fraction y



Temperature T



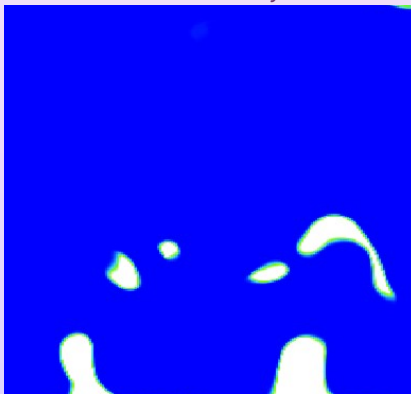
◀ Geometry

▶ Play

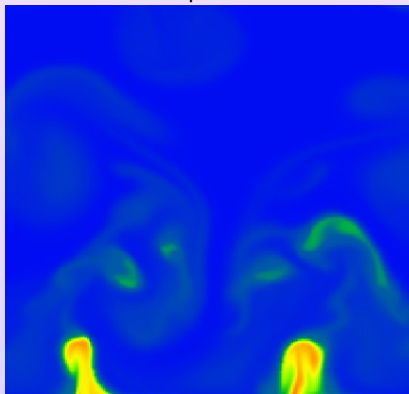
▶▶ Skip

Film

Mass Fraction y



Temperature T



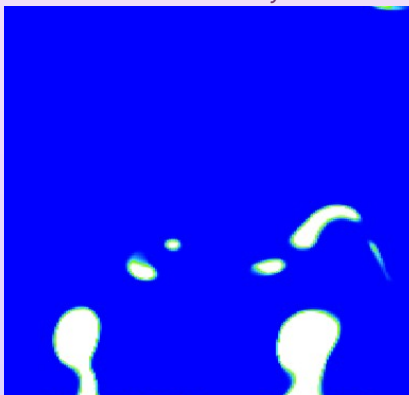
◀ Geometry

▶ Play

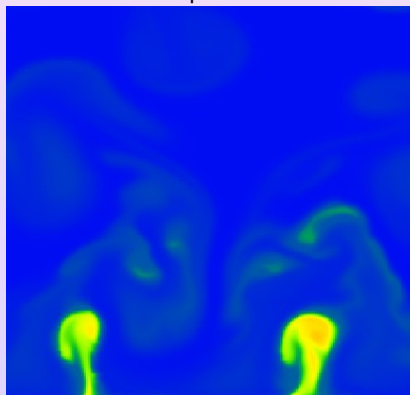
▶▶ Skip

Film

Mass Fraction y



Temperature T



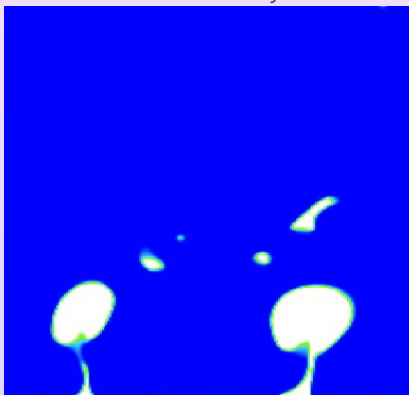
◀ Geometry

▶ Play

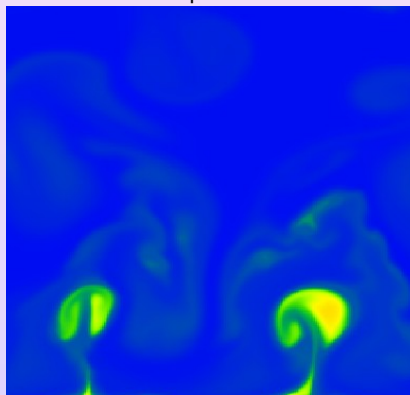
▶▶ Skip

Film

Mass Fraction y



Temperature T



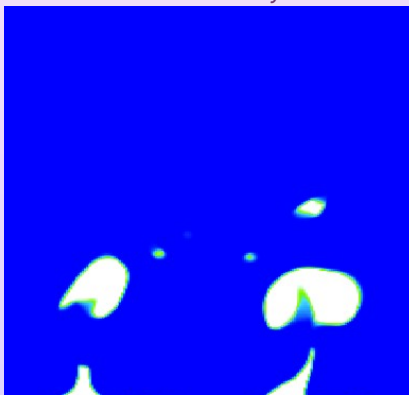
◀ Geometry

▶ Play

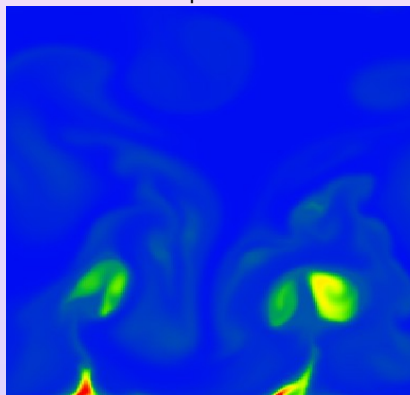
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

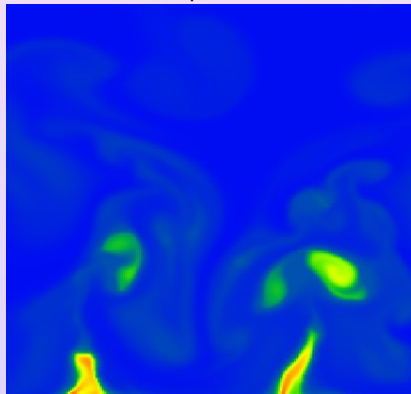
▶▶ Skip

Film

Mass Fraction y



Temperature T



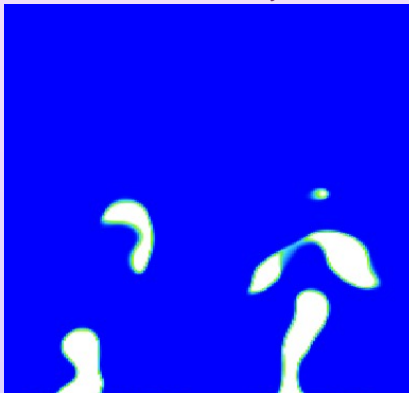
◀ Geometry

▶ Play

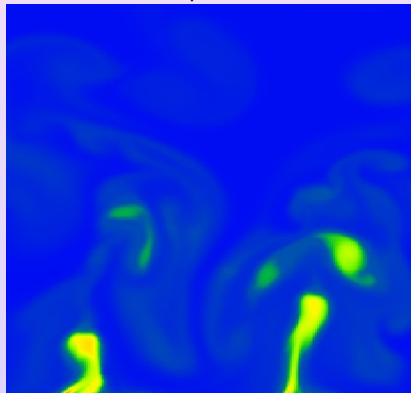
▶▶ Skip

Film

Mass Fraction y



Temperature T



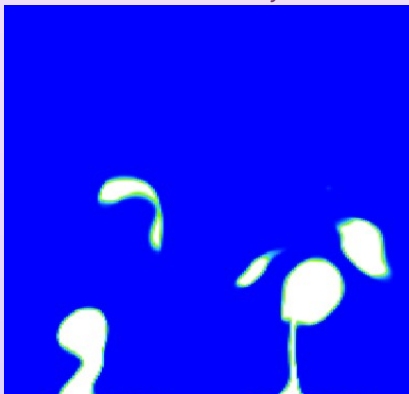
◀ Geometry

▶ Play

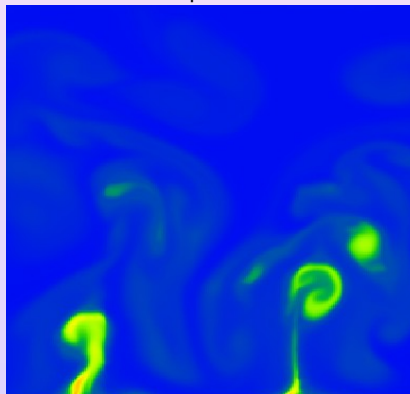
▶▶ Skip

Film

Mass Fraction y



Temperature T



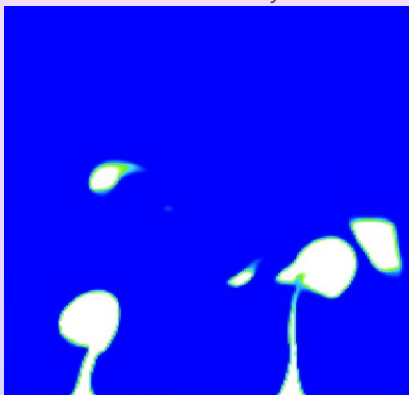
◀ Geometry

▶ Play

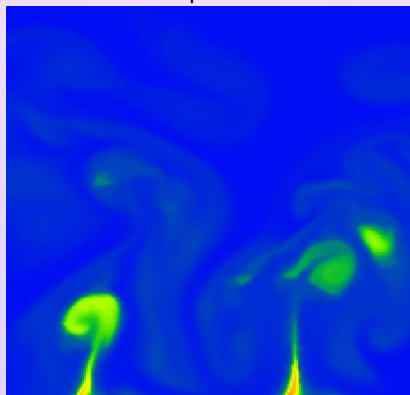
▶▶ Skip

Film

Mass Fraction y



Temperature T



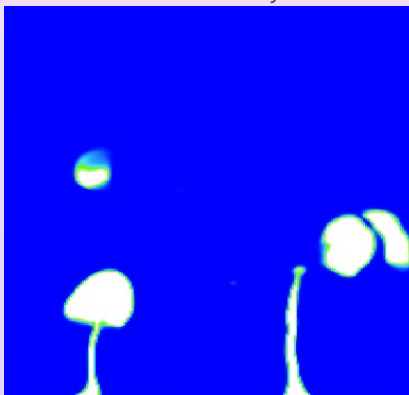
◀ Geometry

▶ Play

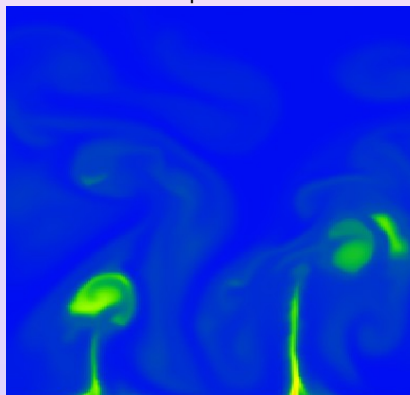
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

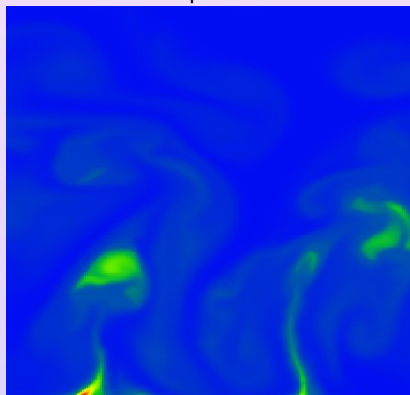
▶▶ Skip

Film

Mass Fraction y



Temperature T



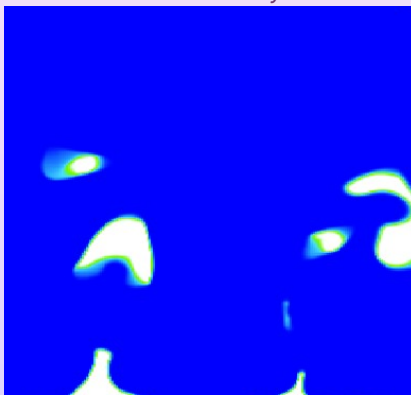
◀ Geometry

▶ Play

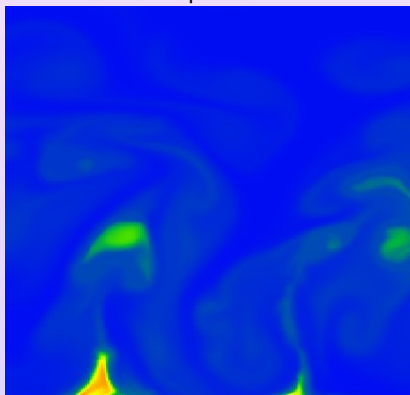
▶▶ Skip

Film

Mass Fraction y



Temperature T



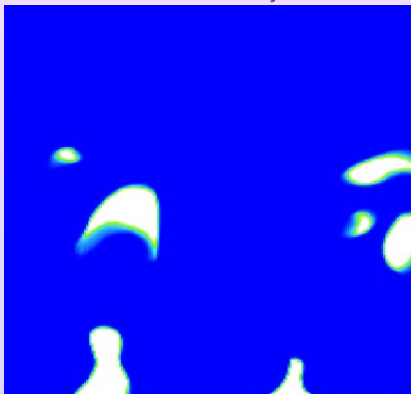
◀ Geometry

▶ Play

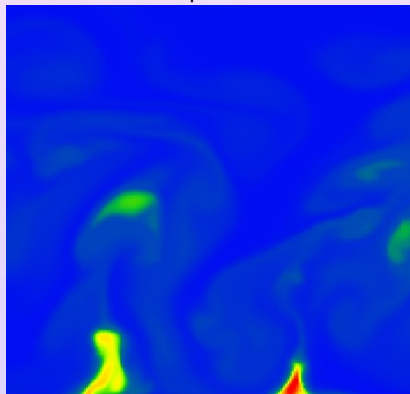
▶▶ Skip

Film

Mass Fraction y



Temperature T



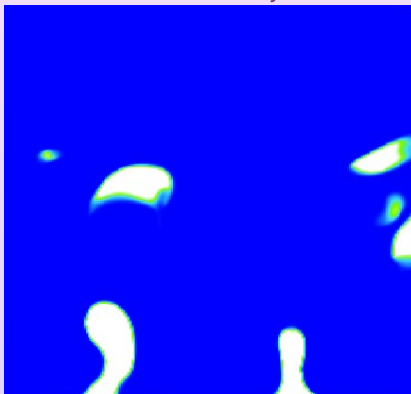
◀ Geometry

▶ Play

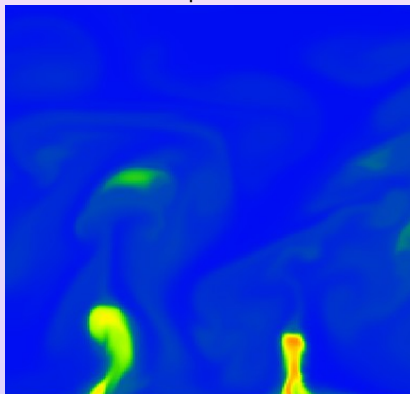
▶▶ Skip

Film

Mass Fraction y



Temperature T



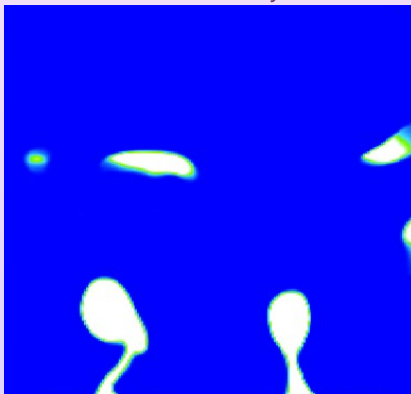
◀ Geometry

▶ Play

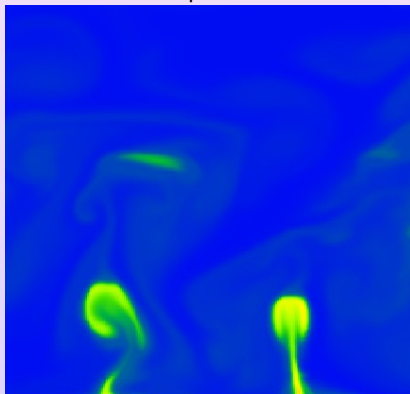
▶▶ Skip

Film

Mass Fraction y



Temperature T



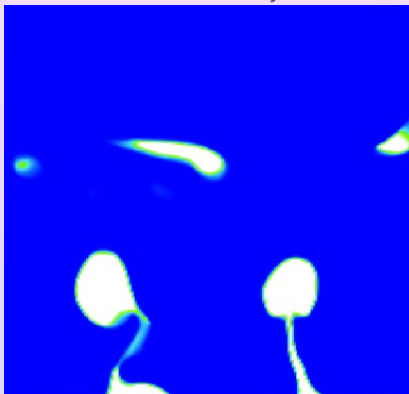
◀ Geometry

▶ Play

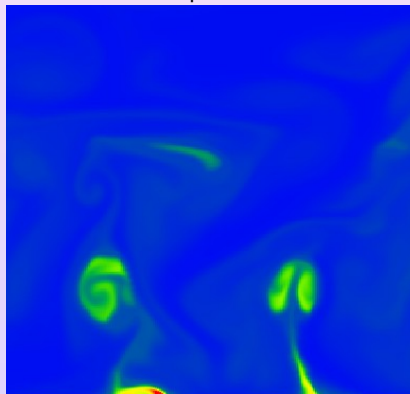
▶▶ Skip

Film

Mass Fraction y



Temperature T



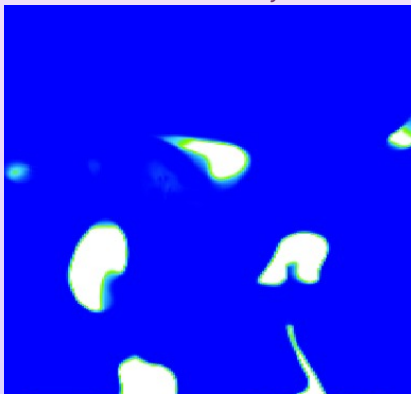
◀ Geometry

▶ Play

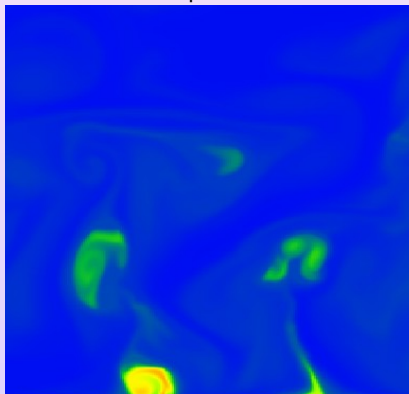
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

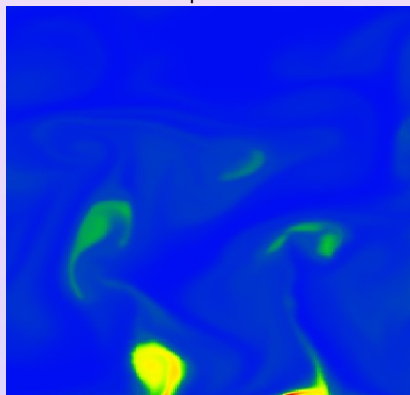
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

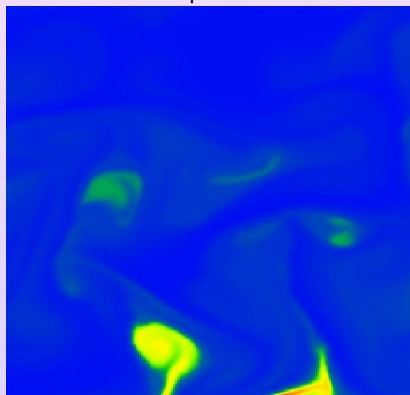
▶▶ Skip

Film

Mass Fraction y



Temperature T



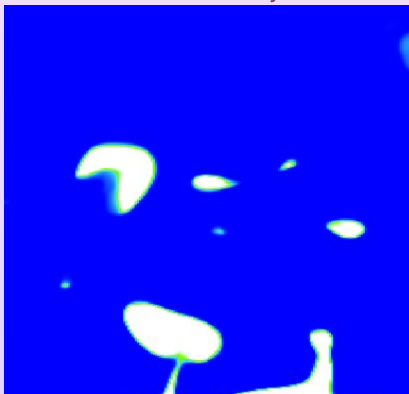
◀ Geometry

▶ Play

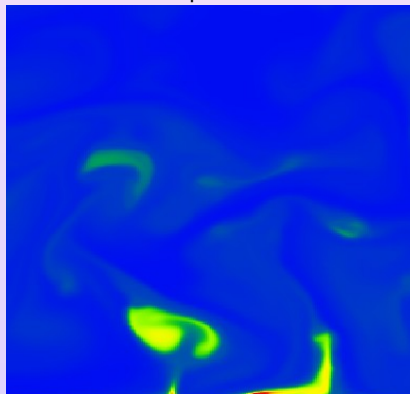
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

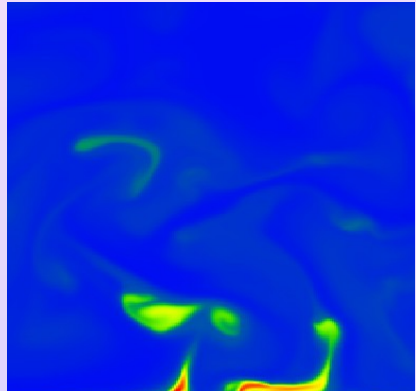
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

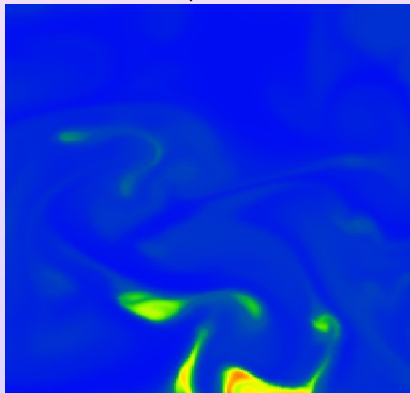
▶▶ Skip

Film

Mass Fraction y



Temperature T



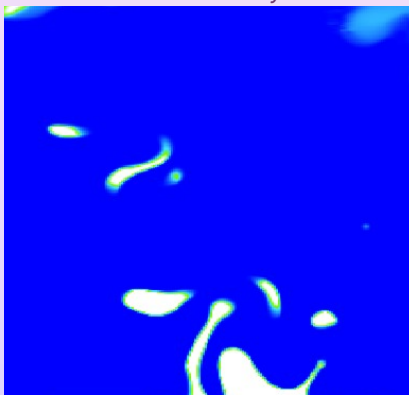
◀ Geometry

▶ Play

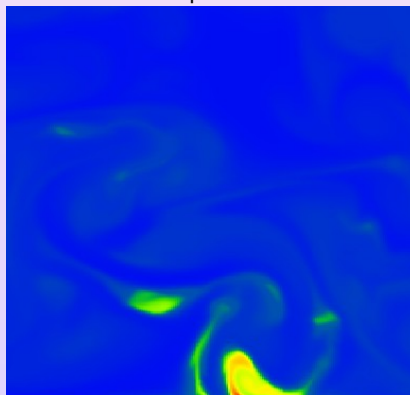
▶▶ Skip

Film

Mass Fraction y



Temperature T



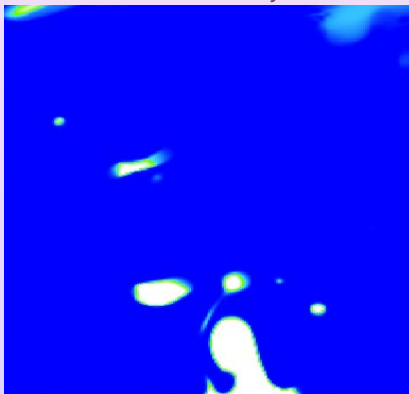
◀ Geometry

▶ Play

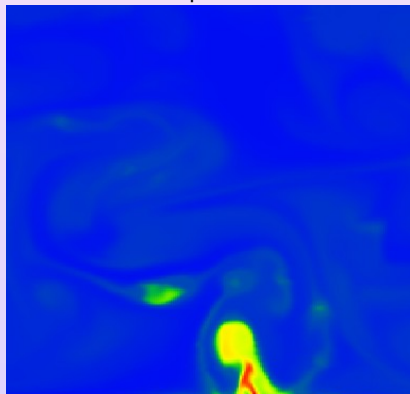
▶▶ Skip

Film

Mass Fraction y



Temperature T



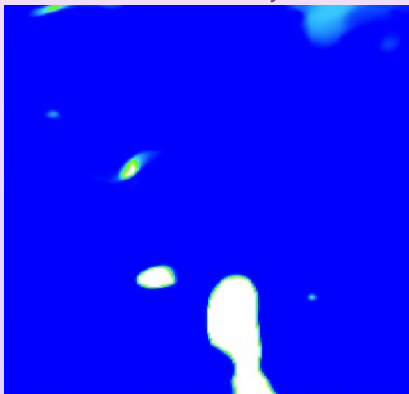
◀ Geometry

▶ Play

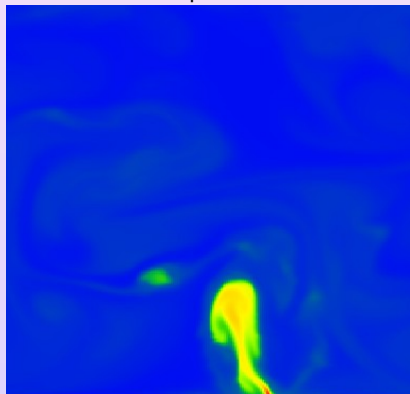
▶▶ Skip

Film

Mass Fraction y



Temperature T



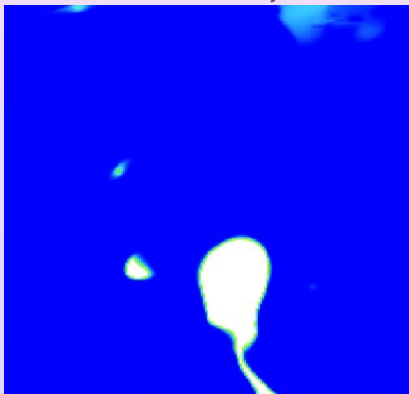
◀ Geometry

▶ Play

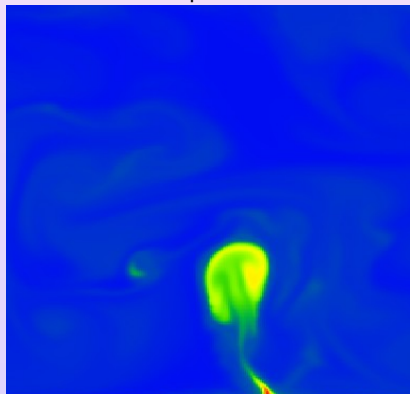
▶▶ Skip

Film

Mass Fraction y



Temperature T



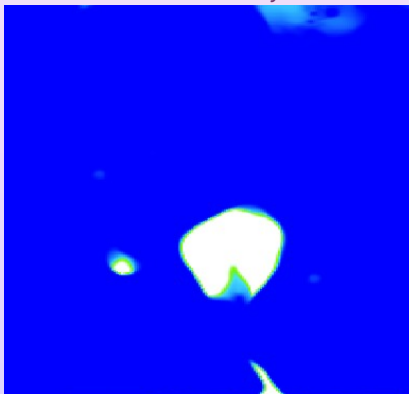
◀ Geometry

▶ Play

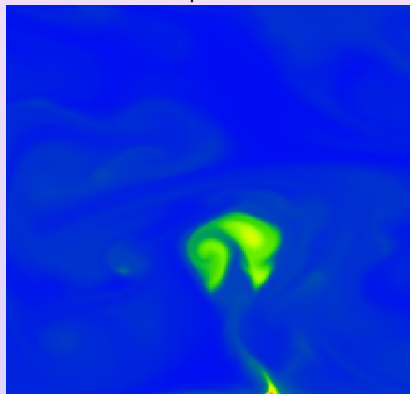
▶▶ Skip

Film

Mass Fraction y



Temperature T



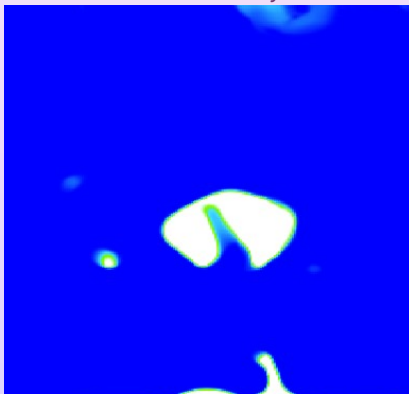
◀ Geometry

▶ Play

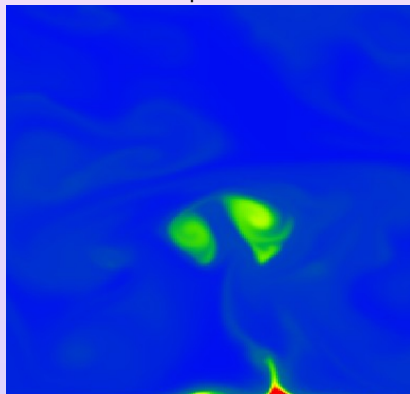
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

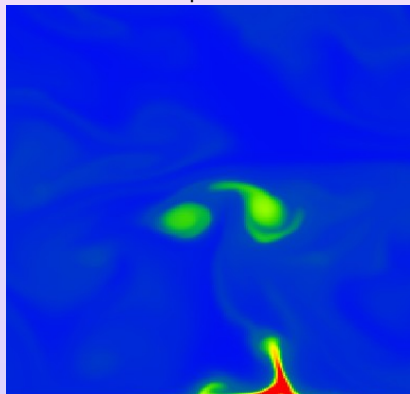
▶▶ Skip

Film

Mass Fraction y



Temperature T



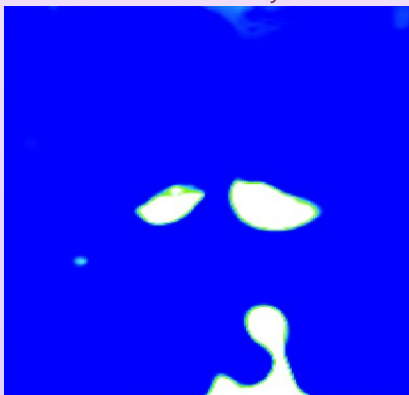
◀ Geometry

▶ Play

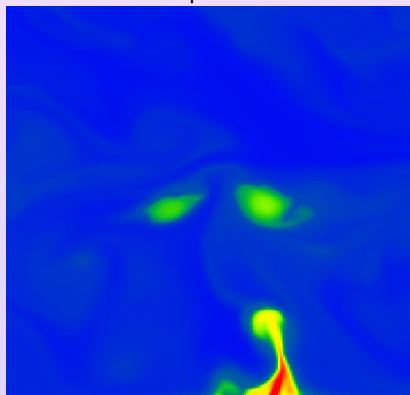
▶▶ Skip

Film

Mass Fraction y



Temperature T



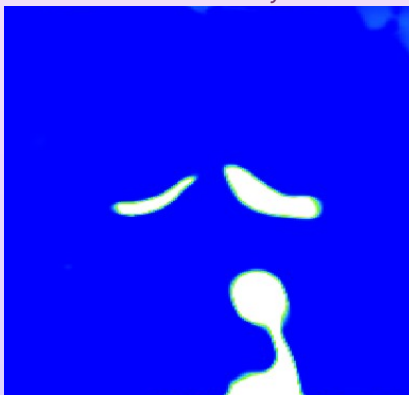
◀ Geometry

▶ Play

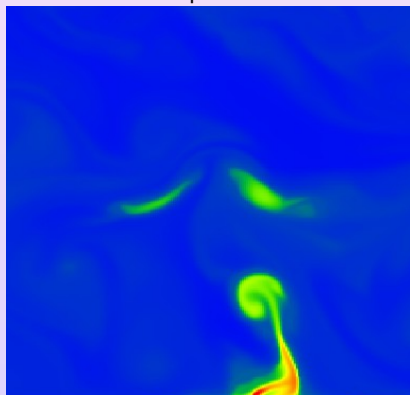
▶▶ Skip

Film

Mass Fraction y



Temperature T



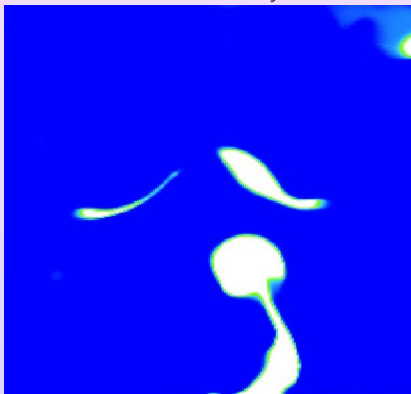
◀ Geometry

▶ Play

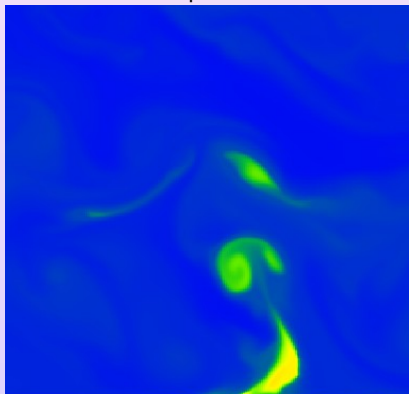
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

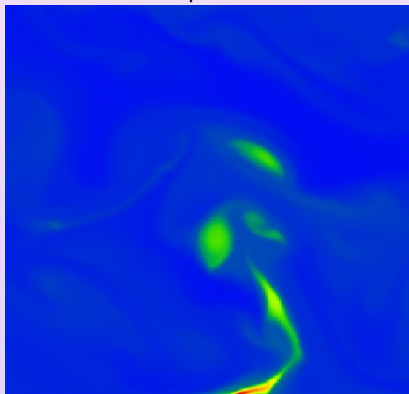
▶▶ Skip

Film

Mass Fraction y



Temperature T



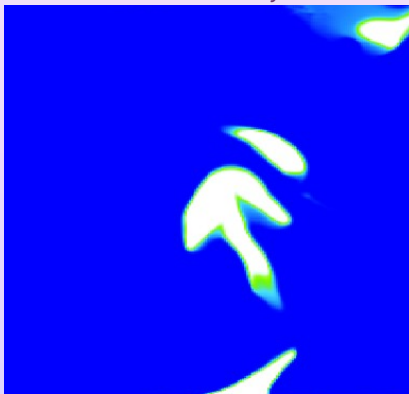
◀ Geometry

▶ Play

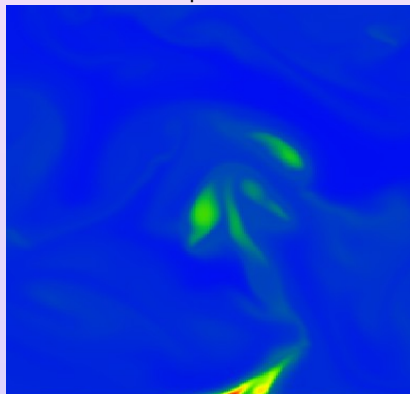
▶▶ Skip

Film

Mass Fraction y



Temperature T



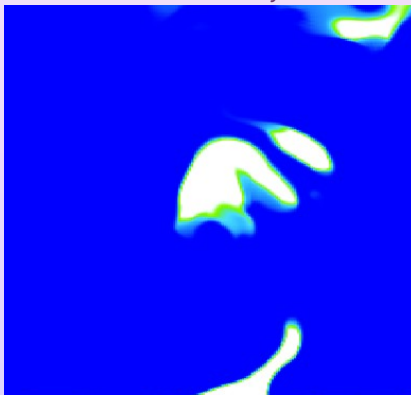
◀ Geometry

▶ Play

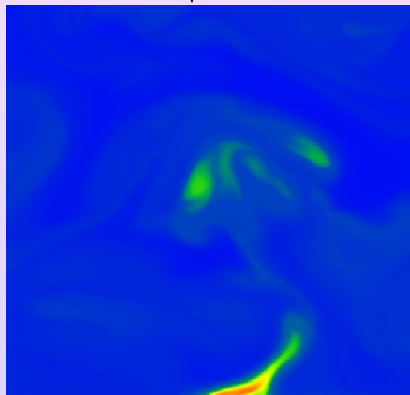
▶▶ Skip

Film

Mass Fraction y



Temperature T



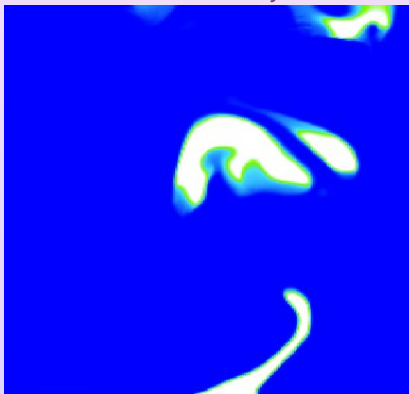
◀ Geometry

▶ Play

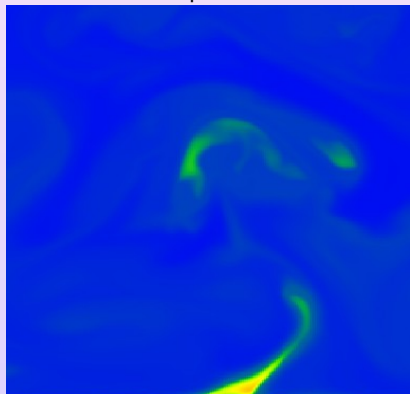
▶▶ Skip

Film

Mass Fraction y



Temperature T



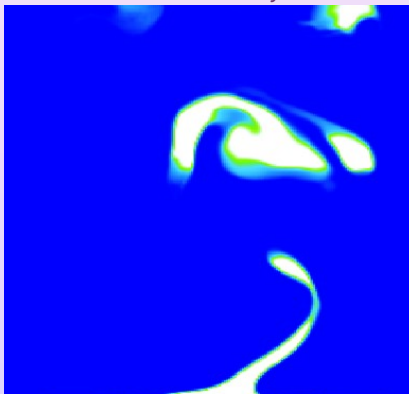
◀ Geometry

▶ Play

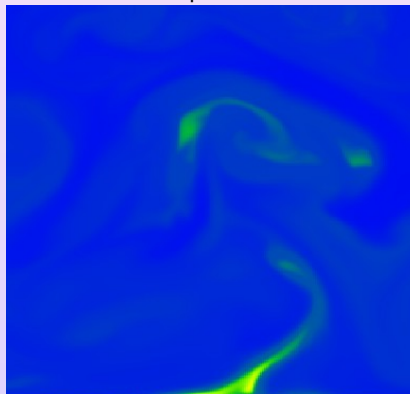
▶▶ Skip

Film

Mass Fraction y



Temperature T



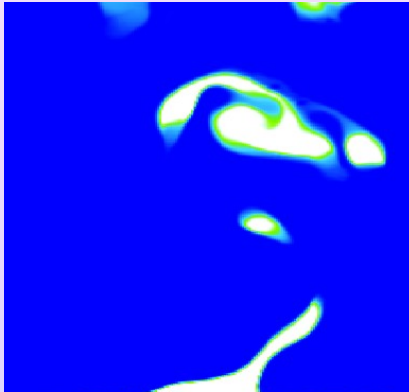
◀ Geometry

▶ Play

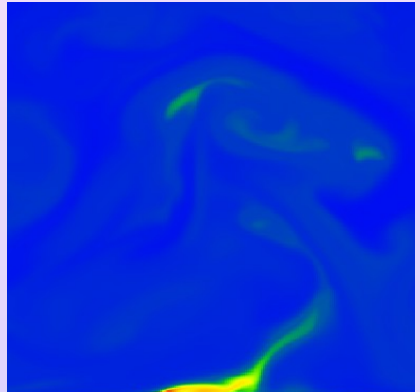
▶▶ Skip

Film

Mass Fraction y



Temperature T



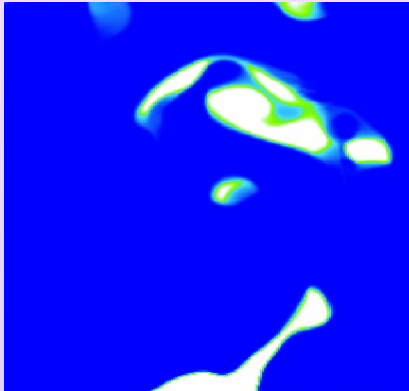
◀ Geometry

▶ Play

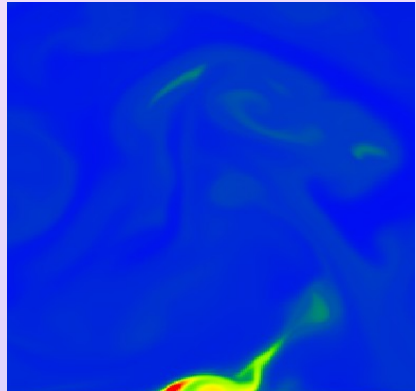
▶▶ Skip

Film

Mass Fraction y



Temperature T



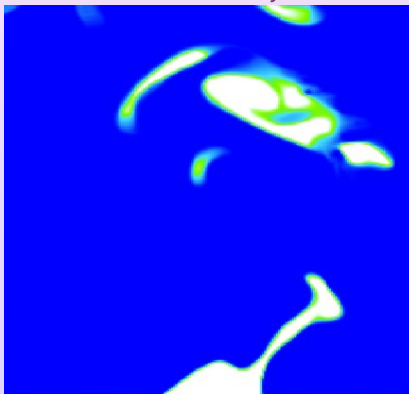
◀ Geometry

▶ Play

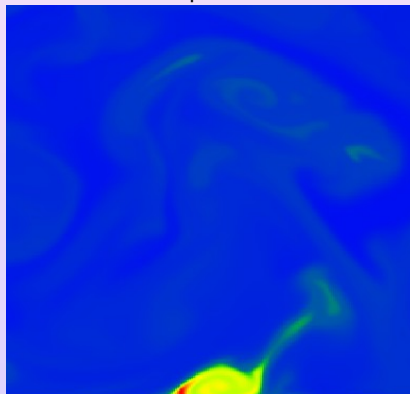
▶▶ Skip

Film

Mass Fraction y



Temperature T



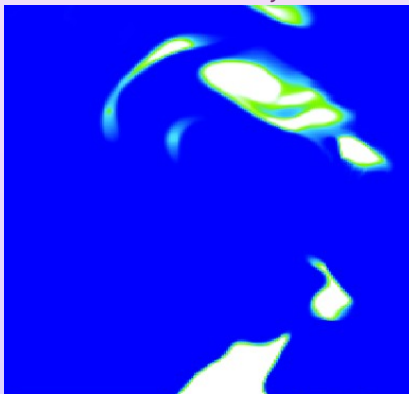
◀ Geometry

▶ Play

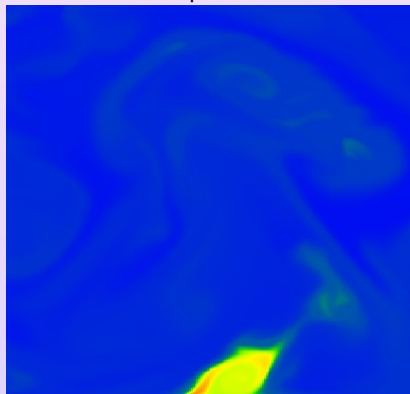
▶▶ Skip

Film

Mass Fraction y



Temperature T



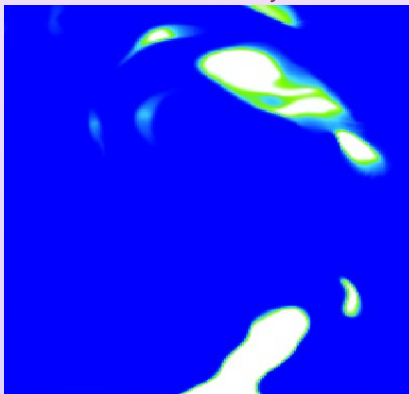
◀ Geometry

▶ Play

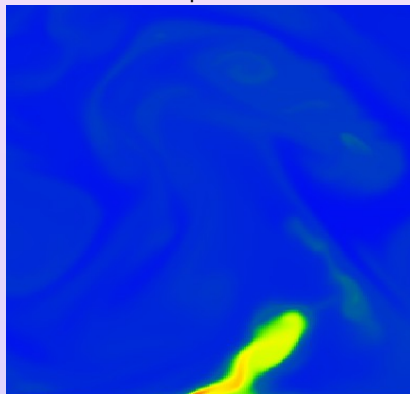
▶▶ Skip

Film

Mass Fraction y



Temperature T



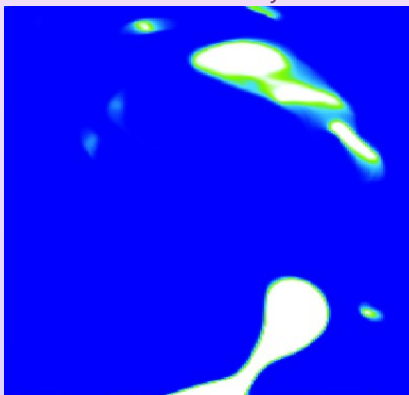
◀ Geometry

▶ Play

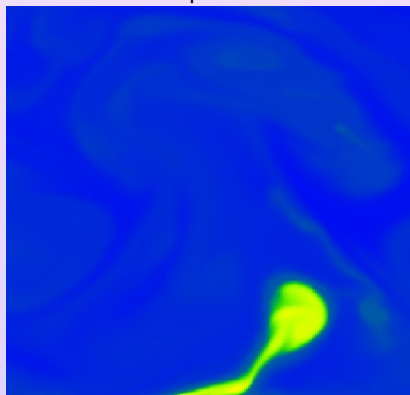
▶▶ Skip

Film

Mass Fraction y



Temperature T



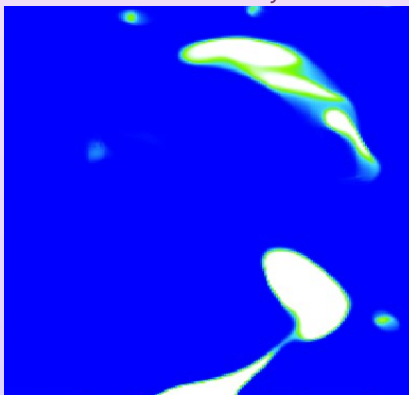
◀ Geometry

▶ Play

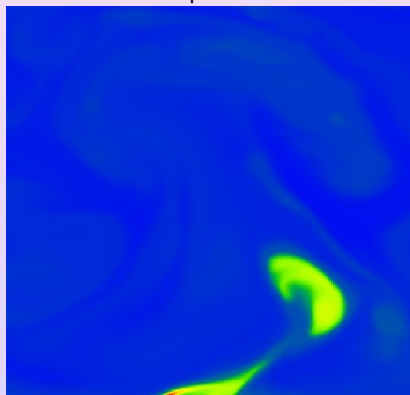
▶▶ Skip

Film

Mass Fraction y



Temperature T



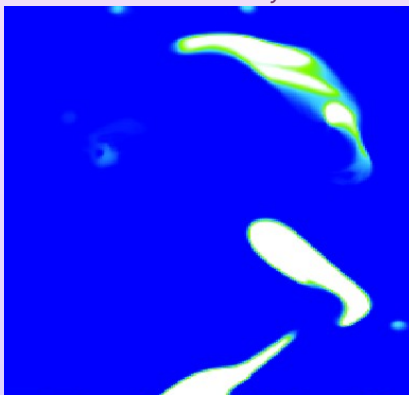
◀ Geometry

▶ Play

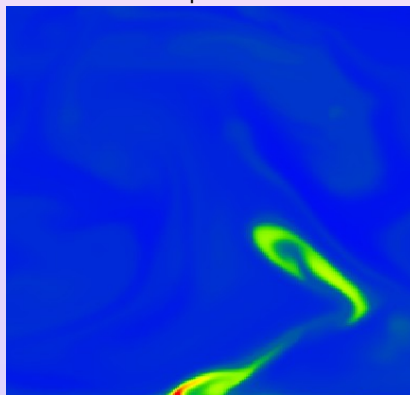
▶▶ Skip

Film

Mass Fraction y



Temperature T



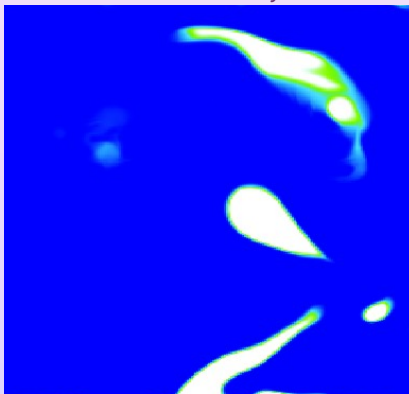
◀ Geometry

▶ Play

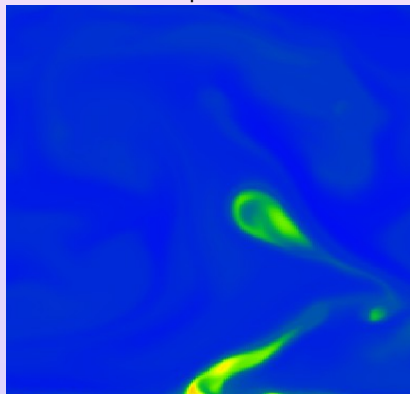
▶▶ Skip

Film

Mass Fraction y



Temperature T



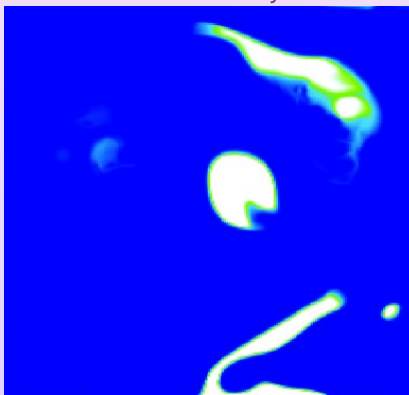
◀ Geometry

▶ Play

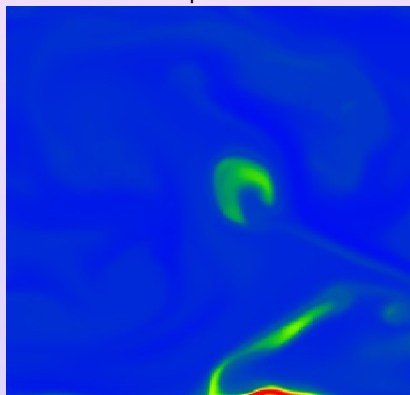
▶▶ Skip

Film

Mass Fraction y



Temperature T



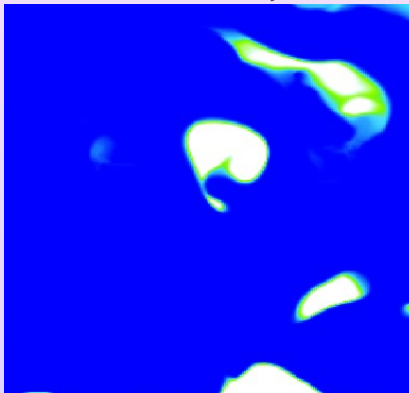
◀ Geometry

▶ Play

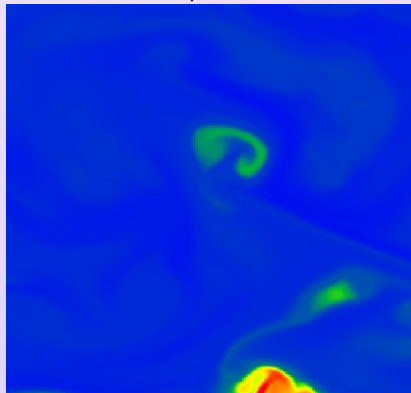
▶▶ Skip

Film

Mass Fraction y



Temperature T



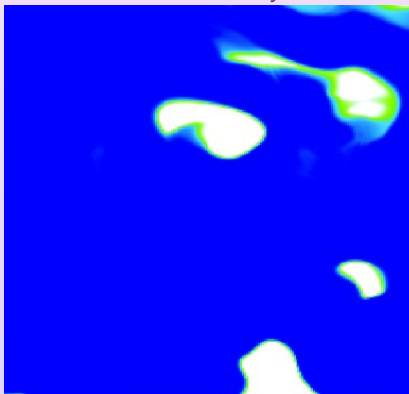
◀ Geometry

▶ Play

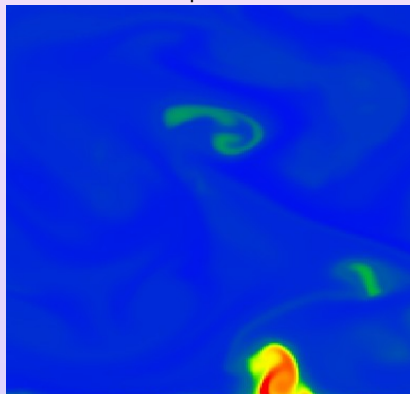
▶▶ Skip

Film

Mass Fraction y



Temperature T



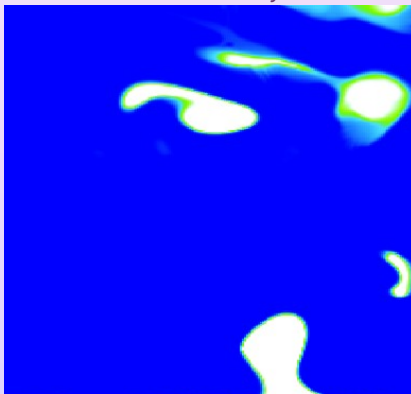
◀ Geometry

▶ Play

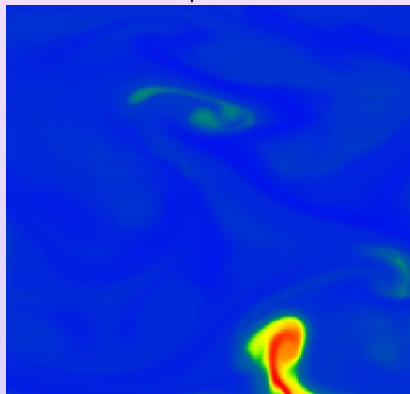
▶▶ Skip

Film

Mass Fraction y



Temperature T



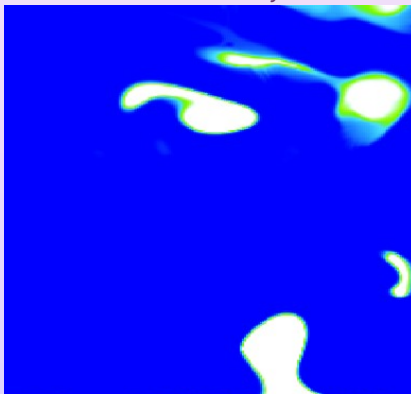
◀ Geometry

▶ Play

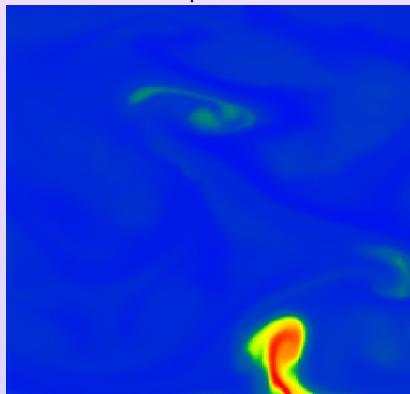
▶▶ Skip

Film

Mass Fraction y



Temperature T



◀ Geometry

▶ Play

▶▶ Skip

Summary

	EOS		Simulation	
	Pure Phases	Equilibrium	Cavitation	Boiling
Virtual Fluid (SG)	✓	✓	✓	✓
Real Fluid (SG)	✓	✓	✓	①
Tabulated	②	✓	③	④

Outline

1 Context

2 Model

- Equation of State WITHOUT Phase Change
- Equation of State WITH Phase Change
- The Phase Change Equation
- Conservation Laws

3 Numerical Approximation

4 Numerical Examples

5 Conclusion

Summary & Perspectives

- Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,

- Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],

- Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

- Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,

- Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],

- Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,

● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],

● Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],

● Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

- Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

- Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✗ **exact Riemann solver for the reference solution (following A. Voss for a Van der Waals EOS)**;
- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],

- Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

- Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

- Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✗ **exact Riemann solver for the reference solution (following A. Voss for a Van der Waals EOS)**;
- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],
- ✗ **operator splitting based on different augmented systems and/or solvers (following R. ABGRALL, R. SAUREL, O. LEMÉTAYER, F. LAGOUTIÈRE, S. KOKH, ...)**,

- Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✗ **exact Riemann solver for the reference solution (following A. Voss for a Van der Waals EOS)**;
- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],
- ✗ **operator splitting based on different augmented systems and/or solvers (following R. ABGRALL, R. SAUREL, O. LEMÉTAYER, F. LAGOUTIÈRE, S. KOKH, ...)**,
- ✗ **no splitting (with the scheme of [M. DUMBSER, C. ENAUX, E. TORO])**;

● Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,

Summary & Perspectives

● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✗ **exact Riemann solver for the reference solution (following A. Voss for a Van der Waals EOS)**;
- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],
- ✗ **operator splitting based on different augmented systems and/or solvers (following R. ABGRALL, R. SAUREL, O. LEMÉTAYER, F. LAGOUTIÈRE, S. KOKH, ...)**,
- ✗ **no splitting (with the scheme of [M. DUMBSER, C. ENAUX, E. TORO])**;

● Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,
- ✗ **2D nucleation with water, dodecane, sodium, ... (implicit transport step)**,

Summary & Perspectives

● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✗ **exact Riemann solver for the reference solution (following A. Voss for a Van der Waals EOS)**;
- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],
- ✗ **operator splitting based on different augmented systems and/or solvers (following R. ABGRALL, R. SAUREL, O. LEMÉTAYER, F. LAGOUTIÈRE, S. KOKH, ...)**,
- ✗ **no splitting (with the scheme of [M. DUMBSER, C. ENAUX, E. TORO])**;

● Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,
- ✗ **2D nucleation with water, dodecane, sodium, ... (implicit transport step)**,
- ✗ **Tabulated EOS**,

Summary & Perspectives

● Diffuse Interface Model

- ✓ general construction of the Equilibrium EOS (also for tabulated data),
- ✓ strict hyperbolicity of the Euler system with the Equilibrium EOS,
- ✗ **Critical Point and Metastability**;

● Numerical Method based on the relaxation approach: augmented systems with relaxation terms

- ✗ **exact Riemann solver for the reference solution (following A. Voss for a Van der Waals EOS)**;
- ✓ operator splitting based on the 5-eqs iso-T with a Roe like solver [G. ALLAIRE, S. CLERC, S. KOKH],
- ✗ **operator splitting based on different augmented systems and/or solvers (following R. ABGRALL, R. SAUREL, O. LEMÉTAYER, F. LAGOUTIÈRE, S. KOKH, ...)**,
- ✗ **no splitting (with the scheme of [M. DUMBSER, C. ENAUX, E. TORO])**;

● Numerical Tests

- ✓ 2D with Stiffened Gas EOS for
 - water for pressure variations,
 - artificial fluid for nucleation,
- ✗ **2D nucleation with water, dodecane, sodium, ... (implicit transport step)**,
- ✗ **Tabulated EOS**,
- ✗ **3D simulations (parallelization)**.

Appendix

- ▶ Stiffened Gas for Water
- ▶ Tabulated EOS for Water
- ▶ Speed of sound
- ▶ Isentropic curves
- ▶ Surface Tension
- ▶ Metastability
- ▶ Critical Point

Stiffened Gas for Water

Phase	c_v [J/(kg·K)]	γ	π [Pa]	q [J/kg]	m [J/(kg·K)]
Water	1816.2	2.35	10^9	-1167.056×10^3	-32765.55596
Steam	1040.14	1.43	0	2030.255×10^3	-33265.65947

Table: Parameters proposed by [O. LE METAYER] for water.

$$(\tau_\alpha, \varepsilon_\alpha) \mapsto s_\alpha = c_{v\alpha} \ln(\varepsilon_\alpha - q_\alpha - \pi_\alpha \tau_\alpha) + c_{v\alpha} (\gamma_\alpha - 1) \ln \tau_\alpha + m_\alpha$$

$$(P, T) \mapsto \varepsilon_\alpha = c_{v\alpha} T \frac{P + \pi_\alpha \gamma_\alpha}{P + \pi_\alpha} + q_\alpha, \quad (P, T) \mapsto \tau_\alpha = c_{v\alpha} (\gamma_\alpha - 1) \frac{T}{P + \pi_\alpha}.$$

$$\left. \begin{array}{l} T^i = 278\text{K} \dots 610\text{K}, \\ g_1(P, T^i) = g_2(P, T^i) \Rightarrow P^{\text{sat}}(T^i) \end{array} \right\} \Rightarrow \mathfrak{A} = \left\{ (T^i, P^{\text{sat}}(T^i)) \right\}_{i=0}^{83}$$

\widehat{P}^{sat} defined by using a least square approximation of \mathfrak{A} :

$$T \mapsto P^{\text{sat}}(T) \approx \widehat{P}^{\text{sat}}(T) \stackrel{\text{def}}{=} \exp \left(\sum_{k=-8}^{k=8} a_k T^k \right)$$

Stiffened Gas for Water

Phase	c_v [J/(kg·K)]	γ	π [Pa]	q [J/kg]	m [J/(kg·K)]
Water	1816.2	2.35	10^9	-1167.056×10^3	-32765.55596
Steam	1040.14	1.43	0	2030.255×10^3	-33265.65947

Table: Parameters proposed by [O. LE METAYER] for water.

$$(\tau_\alpha, \varepsilon_\alpha) \mapsto s_\alpha = c_{v\alpha} \ln(\varepsilon_\alpha - q_\alpha - \pi_\alpha \tau_\alpha) + c_{v\alpha} (\gamma_\alpha - 1) \ln \tau_\alpha + m_\alpha$$

$$(P, T) \mapsto \varepsilon_\alpha = c_{v\alpha} T \frac{P + \pi_\alpha \gamma_\alpha}{P + \pi_\alpha} + q_\alpha, \quad (P, T) \mapsto \tau_\alpha = c_{v\alpha} (\gamma_\alpha - 1) \frac{T}{P + \pi_\alpha}.$$

$$\left. \begin{array}{l} T^i = 278\text{K} \dots 610\text{K}, \\ g_1(P, T^i) = g_2(P, T^i) \Rightarrow P^{\text{sat}}(T^i) \end{array} \right\} \Rightarrow \mathfrak{A} = \left\{ (T^i, P^{\text{sat}}(T^i)) \right\}_{i=0}^{83}$$

\widehat{P}^{sat} defined by using a least square approximation of \mathfrak{A} :

$$T \mapsto P^{\text{sat}}(T) \approx \widehat{P}^{\text{sat}}(T) \stackrel{\text{def}}{=} \exp \left(\sum_{k=-8}^{k=8} a_k T^k \right)$$

Stiffened Gas for Water

Phase	c_v [J/(kg·K)]	γ	π [Pa]	q [J/kg]	m [J/(kg·K)]
Water	1816.2	2.35	10^9	-1167.056×10^3	-32765.55596
Steam	1040.14	1.43	0	2030.255×10^3	-33265.65947

Table: Parameters proposed by [O. LE METAYER] for water.

$$(\tau_\alpha, \varepsilon_\alpha) \mapsto s_\alpha = c_{v\alpha} \ln(\varepsilon_\alpha - q_\alpha - \pi_\alpha \tau_\alpha) + c_{v\alpha} (\gamma_\alpha - 1) \ln \tau_\alpha + m_\alpha$$

$$(P, T) \mapsto \varepsilon_\alpha = c_{v\alpha} T \frac{P + \pi_\alpha \gamma_\alpha}{P + \pi_\alpha} + q_\alpha, \quad (P, T) \mapsto \tau_\alpha = c_{v\alpha} (\gamma_\alpha - 1) \frac{T}{P + \pi_\alpha}.$$

$$\left. \begin{array}{l} T^i = 278\text{K} \dots 610\text{K}, \\ g_1(P, T^i) = g_2(P, T^i) \Rightarrow P^{\text{sat}}(T^i) \end{array} \right\} \Rightarrow \mathfrak{A} = \left\{ (T^i, P^{\text{sat}}(T^i)) \right\}_{i=0}^{83}$$

\widehat{P}^{sat} defined by using a least square approximation of \mathfrak{A} :

$$T \mapsto P^{\text{sat}}(T) \approx \widehat{P}^{\text{sat}}(T) \stackrel{\text{def}}{=} \exp \left(\sum_{k=-8}^{k=8} a_k T^k \right)$$

Stiffened Gas for Water

Phase	c_v [J/(kg·K)]	γ	π [Pa]	q [J/kg]	m [J/(kg·K)]
Water	1816.2	2.35	10^9	-1167.056×10^3	-32765.55596
Steam	1040.14	1.43	0	2030.255×10^3	-33265.65947

Table: Parameters proposed by [O. LE METAYER] for water.

$$(\tau_\alpha, \varepsilon_\alpha) \mapsto s_\alpha = c_{v\alpha} \ln(\varepsilon_\alpha - q_\alpha - \pi_\alpha \tau_\alpha) + c_{v\alpha} (\gamma_\alpha - 1) \ln \tau_\alpha + m_\alpha$$

$$(P, T) \mapsto \varepsilon_\alpha = c_{v\alpha} T \frac{P + \pi_\alpha \gamma_\alpha}{P + \pi_\alpha} + q_\alpha, \quad (P, T) \mapsto \tau_\alpha = c_{v\alpha} (\gamma_\alpha - 1) \frac{T}{P + \pi_\alpha}.$$

$$\left. \begin{array}{l} T^i = 278\text{K} \dots 610\text{K}, \\ g_1(P, T^i) = g_2(P, T^i) \Rightarrow P^{\text{sat}}(T^i) \end{array} \right\} \Rightarrow \mathfrak{A} = \left\{ (T^i, P^{\text{sat}}(T^i)) \right\}_{i=0}^{83}$$

\widehat{P}^{sat} defined by using a least square approximation of \mathfrak{A} :

$$T \mapsto P^{\text{sat}}(T) \approx \widehat{P}^{\text{sat}}(T) \stackrel{\text{def}}{=} \exp \left(\sum_{k=-8}^{k=8} a_k T^k \right)$$

Water Tabulated EOS

$$\left. \begin{array}{l} T^i = 278\text{K} \dots 610\text{K}, \\ \varepsilon_{\alpha}^{\text{sat}}(T^i), \tau_{\alpha}^{\text{sat}}(T^i) \text{ found in the tables} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \mathfrak{A} = \left\{ \left(T_i, \frac{1}{\varepsilon_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \\ \mathfrak{B} = \left\{ \left(T_i, \frac{\varepsilon_{\text{liq}}^{\text{sat}}(T_i)}{\varepsilon_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \\ \mathfrak{C} = \left\{ \left(T_i, \frac{1}{\tau_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \\ \mathfrak{D} = \left\{ \left(T_i, \frac{\tau_{\text{liq}}^{\text{sat}}(T_i)}{\tau_{\text{vap}}^{\text{sat}}(T_i)} \right) \right\}_i \end{array} \right.$$

$\widehat{\varepsilon}_{\alpha}^{\text{sat}}$ and $\widehat{\tau}_{\alpha}^{\text{sat}}$ defined by using a least square approximation of \mathfrak{A} , \mathfrak{B} , \mathfrak{C} and \mathfrak{D} :

$$T \mapsto \varepsilon_{\text{vap}}^{\text{sat}} \approx \widehat{\varepsilon}_{\text{vap}}^{\text{sat}} \stackrel{\text{def}}{=} \frac{1}{\sum_{k=0}^6 a_k T^k}$$

$$T \mapsto \varepsilon_{\text{liq}}^{\text{sat}} \approx \widehat{\varepsilon}_{\text{liq}}^{\text{sat}} \stackrel{\text{def}}{=} \widehat{\varepsilon}_{\text{vap}}^{\text{sat}}(T) \sum_{k=0}^6 b_k T^k$$

$$T \mapsto \tau_{\text{vap}}^{\text{sat}} \approx \widehat{\tau}_{\text{vap}}^{\text{sat}} \stackrel{\text{def}}{=} \frac{1}{\sum_{k=0}^8 c_k T^k}$$

$$T \mapsto \tau_{\text{liq}}^{\text{sat}} \approx \widehat{\tau}_{\text{liq}}^{\text{sat}} \stackrel{\text{def}}{=} \widehat{\tau}_{\text{vap}}^{\text{sat}}(T) \sum_{k=0}^9 d_k T^k$$

Speed of sound

$$c^2 \stackrel{\text{def}}{=} \tau^2 \left(P^{\text{eq}} \frac{\partial P^{\text{eq}}}{\partial \varepsilon} \Big|_{\tau} - \frac{\partial P^{\text{eq}}}{\partial \tau} \Big|_{\varepsilon} \right) = \overset{0}{-\tau^2 T^{\text{eq}}} \begin{bmatrix} P^{\text{eq}}, & -1 \end{bmatrix} \begin{bmatrix} S_{\varepsilon\varepsilon}^{\text{eq}} & S_{\tau\varepsilon}^{\text{eq}} \\ S_{\tau\varepsilon}^{\text{eq}} & S_{\tau\tau}^{\text{eq}} \end{bmatrix} \begin{bmatrix} P^{\text{eq}} \\ -1 \end{bmatrix} \leq 0$$

HESSIAN MATRIX OF $\mathbf{w} \mapsto s^{\text{eq}}$

- for all \mathbf{w} pure phase state

$$\mathbf{v}^T d^2 s^{\text{eq}}(\mathbf{w}) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

- for all \mathbf{w} equilibrium mixture state

$$\exists \mathbf{v}(\mathbf{w}) \neq 0 \text{ s.t. } (\mathbf{v}(\mathbf{w}))^T d^2 s^{\text{eq}}(\mathbf{w}) \mathbf{v}(\mathbf{w}) = 0.$$

Speed of sound

$$c^2 \stackrel{\text{def}}{=} \tau^2 \left(P^{\text{eq}} \frac{\partial P^{\text{eq}}}{\partial \varepsilon} \Big|_{\tau} - \frac{\partial P^{\text{eq}}}{\partial \tau} \Big|_{\varepsilon} \right) = \overset{0}{-\tau^2 T^{\text{eq}}} \begin{bmatrix} P^{\text{eq}}, & -1 \end{bmatrix} \begin{bmatrix} S_{\varepsilon\varepsilon}^{\text{eq}} & S_{\tau\varepsilon}^{\text{eq}} \\ S_{\tau\varepsilon}^{\text{eq}} & S_{\tau\tau}^{\text{eq}} \end{bmatrix} \begin{bmatrix} P^{\text{eq}} \\ -1 \end{bmatrix} \leq 0$$

HESSIAN MATRIX OF $\mathbf{w} \mapsto s^{\text{eq}}$

- for all \mathbf{w} pure phase state

$$\mathbf{v}^T d^2 s^{\text{eq}}(\mathbf{w}) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

- for all \mathbf{w} equilibrium mixture state

$$\exists \mathbf{v}(\mathbf{w}) \neq 0 \text{ s.t. } (\mathbf{v}(\mathbf{w}))^T d^2 s^{\text{eq}}(\mathbf{w}) \mathbf{v}(\mathbf{w}) = 0.$$

Speed of sound

$$c^2 \stackrel{\text{def}}{=} \tau^2 \left(P^{\text{eq}} \frac{\partial P^{\text{eq}}}{\partial \varepsilon} \Big|_{\tau} - \frac{\partial P^{\text{eq}}}{\partial \tau} \Big|_{\varepsilon} \right) = \overset{0}{-\tau^2 T^{\text{eq}}} \begin{bmatrix} P^{\text{eq}}, & -1 \end{bmatrix} \begin{bmatrix} S_{\varepsilon\varepsilon}^{\text{eq}} & S_{\tau\varepsilon}^{\text{eq}} \\ S_{\tau\varepsilon}^{\text{eq}} & S_{\tau\tau}^{\text{eq}} \end{bmatrix} \begin{bmatrix} P^{\text{eq}} \\ -1 \end{bmatrix} \leq 0$$

HESSIAN MATRIX OF $\mathbf{w} \mapsto S^{\text{eq}}$

- for all \mathbf{w} pure phase state

$$\mathbf{v}^T d^2 S^{\text{eq}}(\mathbf{w}) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

- for all \mathbf{w} equilibrium mixture state

$$\exists \mathbf{v}(\mathbf{w}) \neq 0 \text{ s.t. } (\mathbf{v}(\mathbf{w}))^T d^2 S^{\text{eq}}(\mathbf{w}) \mathbf{v}(\mathbf{w}) = 0.$$

Speed of sound

$$c^2 \stackrel{\text{def}}{=} \tau^2 \left(P^{\text{eq}} \frac{\partial P^{\text{eq}}}{\partial \varepsilon} \Big|_{\tau} - \frac{\partial P^{\text{eq}}}{\partial \tau} \Big|_{\varepsilon} \right) = \overset{0}{-\tau^2 T^{\text{eq}}} \begin{bmatrix} P^{\text{eq}}, & -1 \end{bmatrix} \begin{bmatrix} S_{\varepsilon\varepsilon}^{\text{eq}} & S_{\tau\varepsilon}^{\text{eq}} \\ S_{\tau\varepsilon}^{\text{eq}} & S_{\tau\tau}^{\text{eq}} \end{bmatrix} \begin{bmatrix} P^{\text{eq}} \\ -1 \end{bmatrix} \leq 0$$

HESSIAN MATRIX OF $\mathbf{w} \mapsto S^{\text{eq}}$

- for all \mathbf{w} pure phase state

$$\mathbf{v}^T d^2 S^{\text{eq}}(\mathbf{w}) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

- for all \mathbf{w} equilibrium mixture state

$$\exists \mathbf{v}(\mathbf{w}) \neq 0 \text{ s.t. } (\mathbf{v}(\mathbf{w}))^T d^2 S^{\text{eq}}(\mathbf{w}) \mathbf{v}(\mathbf{w}) = 0.$$

$$\forall \mathbf{w} \text{ equilibrium mixture state, } \mathbf{v}(\mathbf{w}) \stackrel{?}{=} [P^{\text{eq}}(\mathbf{w}), -1]$$

Speed of sound

$$c^2 \stackrel{\text{def}}{=} \tau^2 \left(P^{\text{eq}} \frac{\partial P^{\text{eq}}}{\partial \varepsilon} \Big|_{\tau} - \frac{\partial P^{\text{eq}}}{\partial \tau} \Big|_{\varepsilon} \right) = \overset{0}{-\tau^2 T^{\text{eq}}} \begin{bmatrix} P^{\text{eq}}, & -1 \end{bmatrix} \begin{bmatrix} S_{\varepsilon\varepsilon}^{\text{eq}} & S_{\tau\varepsilon}^{\text{eq}} \\ S_{\tau\varepsilon}^{\text{eq}} & S_{\tau\tau}^{\text{eq}} \end{bmatrix} \begin{bmatrix} P^{\text{eq}} \\ -1 \end{bmatrix} \leq 0$$

HESSIAN MATRIX OF $\mathbf{w} \mapsto S^{\text{eq}}$

- for all \mathbf{w} pure phase state

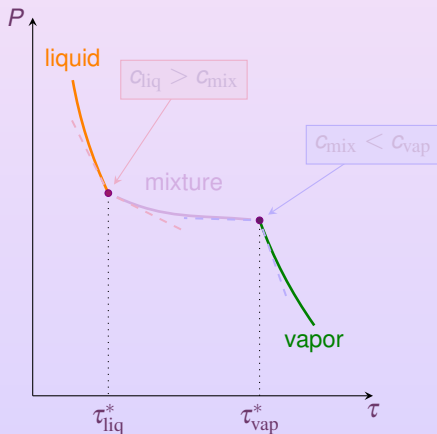
$$\mathbf{v}^T d^2 S^{\text{eq}}(\mathbf{w}) \mathbf{v} < 0 \quad \forall \mathbf{v} \neq 0,$$

- for all \mathbf{w} equilibrium mixture state

$$\exists \mathbf{v}(\mathbf{w}) \neq 0 \text{ s.t. } (\mathbf{v}(\mathbf{w}))^T d^2 S^{\text{eq}}(\mathbf{w}) \mathbf{v}(\mathbf{w}) = 0.$$

$$\forall \mathbf{w} \text{ equilibrium mixture state, } \mathbf{v}(\mathbf{w}) \not\propto [P^{\text{eq}}(\mathbf{w}), -1]$$

Isentropic curves



$$\gamma \stackrel{\text{def}}{=} - \frac{\tau}{P} \frac{\partial P}{\partial \tau} \Big|_s$$

$$\Gamma \stackrel{\text{def}}{=} \tau \frac{\partial P}{\partial \varepsilon} \Big|_{\tau}$$

$$\mathcal{G} \stackrel{\text{def}}{=} \frac{\tau^2}{2\gamma P} \frac{\partial^2 P}{\partial \tau^2} \Big|_s$$

● Pure Phases

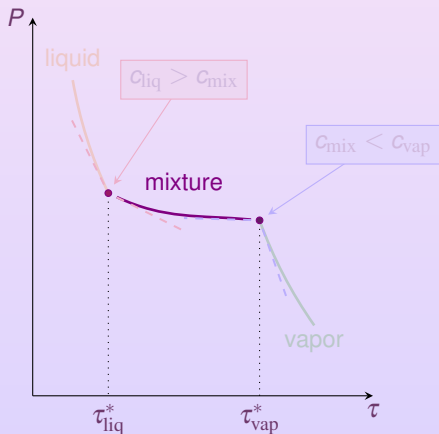
- (H) $\gamma > 0$
- (H) $\Gamma > 0$
- (H) $\mathcal{G} > 0$

● Mixture

- (P) $\gamma > 0$
- (P) $\Gamma > 0$
- (H) $\mathcal{G} > 0$

- Regularity: [J. CORREIA, P.G. LEFLOCH, M.D. THANH]
- Loss of convexity: [A. VOSS]

Isentropic curves



$$\gamma \stackrel{\text{def}}{=} - \frac{\tau}{P} \frac{\partial P}{\partial \tau} \Big|_s$$

$$\Gamma \stackrel{\text{def}}{=} \tau \frac{\partial P}{\partial \varepsilon} \Big|_{\tau}$$

$$\mathcal{G} \stackrel{\text{def}}{=} \frac{\tau^2}{2\gamma P} \frac{\partial^2 P}{\partial \tau^2} \Big|_s$$

● Pure Phases

- (H) $\gamma > 0$
- (H) $\Gamma > 0$
- (H) $\mathcal{G} > 0$

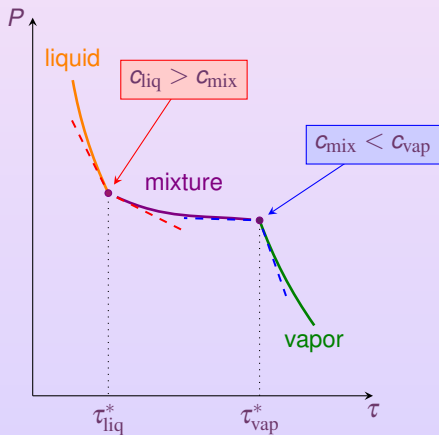
● Mixture

- (P) $\gamma > 0$
- (P) $\Gamma > 0$
- (H) $\mathcal{G} > 0$

● **Regularity:** [J. CORREIA, P.G. LEFLOCH, M.D. THANH]

● **Loss of convexity:** [A. VOSS]

Isentropic curves



$$\gamma \stackrel{\text{def}}{=} - \frac{\tau}{P} \frac{\partial P}{\partial \tau} \Big|_s$$

$$\Gamma \stackrel{\text{def}}{=} \tau \frac{\partial P}{\partial \varepsilon} \Big|_{\tau}$$

$$\mathcal{G} \stackrel{\text{def}}{=} \frac{\tau^2}{2\gamma P} \frac{\partial^2 P}{\partial \tau^2} \Big|_s$$

- Pure Phases

- (H) $\gamma > 0$
- (H) $\Gamma > 0$
- (H) $\mathcal{G} > 0$

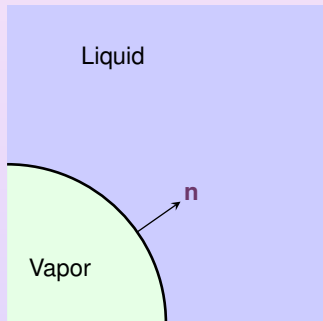
- Mixture

- (P) $\gamma > 0$
- (P) $\Gamma > 0$
- (H) $\mathcal{G} > 0$

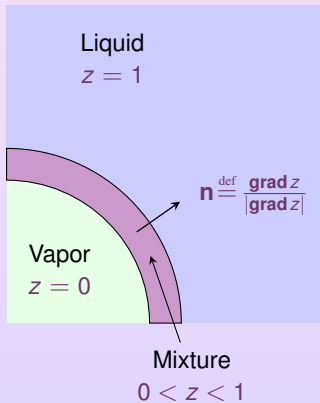
- Regularity: [J. CORREIA, P.G. LEFLOCH, M.D. THANH]
- Loss of convexity: [A. VOSS]

Continuum Surface Force (CSF) Approach

Physical Interface

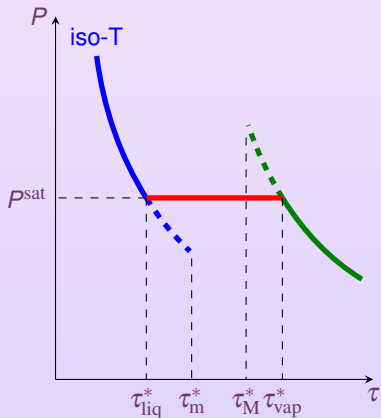


Diffuse Interface



$$\Pi_{\text{tension}} = -\sigma \text{div}(\mathbf{n})\mathbf{n}$$

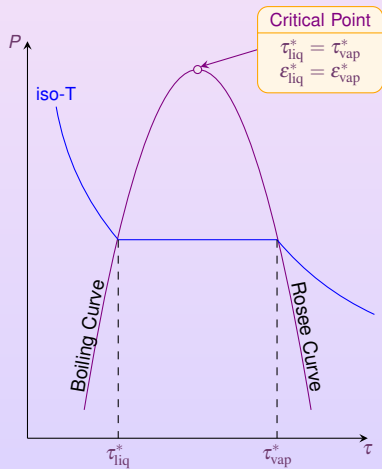
Metastability



$$P^{\text{eq}} = \begin{cases} P_{\text{liq}}, & \text{if } \tau < \tau_{\text{liq}}^*, \\ P^{\text{sat}}, & \text{if } \tau_{\text{liq}}^* < \tau < \tau_{\text{vap}}^*, \\ P_{\text{vap}}, & \text{if } \tau_{\text{vap}}^* < \tau. \end{cases}$$

$$P^{\text{met}} = \begin{cases} P_{\text{liq}}, & \text{if } \tau < \tau_{\text{liq}}^*, \\ [P^{\text{sat}} \text{ or } P_{\text{liq}}], & \text{if } \tau_{\text{liq}}^* < \tau < \tau_m^*, \\ P^{\text{sat}}, & \text{if } \tau_m^* < \tau < \tau_M^*, \\ [P^{\text{sat}} \text{ or } P_{\text{vap}}], & \text{if } \tau_M^* < \tau < \tau_{\text{vap}}^*, \\ P_{\text{vap}}, & \text{if } \tau_{\text{vap}}^* < \tau, \end{cases}$$

Critical Point



PHYSIC

- 2 Pure Phases EOS $(\tau, \epsilon) \mapsto P_\alpha$
 - 1 Saturation EOS $\tau \mapsto P^{\text{sat}}$
- Eq

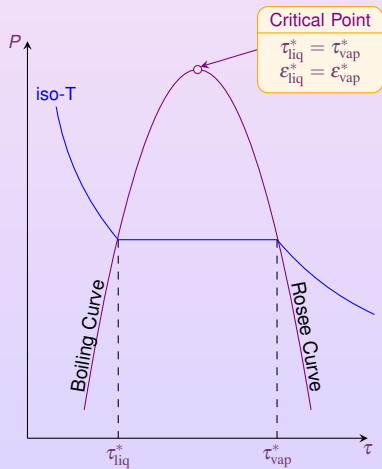
EOS

PG $\epsilon_{\text{liq}}^* = \epsilon_{\text{vap}}^* \Leftrightarrow c_{V\text{liq}} = c_{V\text{vap}}$ (indip. of T)

SG $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\tau_{\text{liq}}^* = \tau_{\text{vap}}^*$ but $\epsilon_{\text{liq}}^* \neq \epsilon_{\text{vap}}^*$

TAB $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\{(\tau, \epsilon_i), (P_\alpha^c)_i\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha$

Critical Point



PHYSIC

- 2 Pure Phases EOS $(\tau, \epsilon) \mapsto P_\alpha$
 - 1 Saturation EOS $\tau \mapsto P^{\text{sat}}$
- Eq

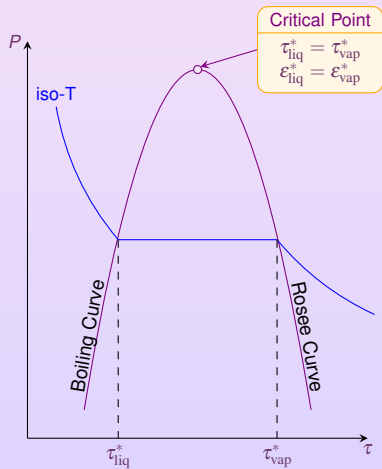
EOS

PG $\epsilon_{\text{liq}}^* = \epsilon_{\text{vap}}^* \Leftrightarrow c_{V\text{liq}} = c_{V\text{vap}}$ (indip. of T)

SG $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\tau_{\text{liq}}^* = \tau_{\text{vap}}^*$ but $\epsilon_{\text{liq}}^* \neq \epsilon_{\text{vap}}^*$

TAB $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\{(\tau, \epsilon)_i, (P_\alpha^c)_i\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha$

Critical Point



PHYSIC

- 2 Pure Phases EOS $(\tau, \epsilon) \mapsto P_\alpha$
 - 1 Saturation EOS $\tau \mapsto P^{\text{sat}}$
- Eq

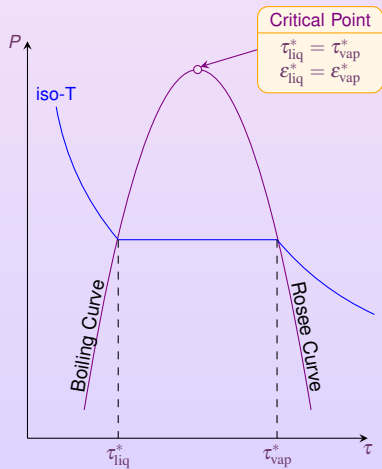
EOS

PG $\epsilon_{\text{liq}}^* = \epsilon_{\text{vap}}^* \Leftrightarrow c_{V\text{liq}} = c_{V\text{vap}}$ (indip. of T)

SG $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\tau_{\text{liq}}^* = \tau_{\text{vap}}^*$ but $\epsilon_{\text{liq}}^* \neq \epsilon_{\text{vap}}^*$

TAB $\{\tau_i, P_i^{\text{sat},e}\}_i \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\{(\tau_i, \epsilon_i), (P_\alpha^c)_i\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha$

Critical Point



PHYSIC

- 2 Pure Phases EOS $(\tau, \epsilon) \mapsto P_\alpha$
 - 1 Saturation EOS $\tau \mapsto P^{\text{sat}}$
- Eq

EOS

PG $\epsilon_{\text{liq}}^* = \epsilon_{\text{vap}}^* \Leftrightarrow c_{V\text{liq}} = c_{V\text{vap}}$ (indip. of T)

SG $\{\tau_i, P_i^{\text{sat,e}}\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\tau_{\text{liq}}^* = \tau_{\text{vap}}^*$ but $\epsilon_{\text{liq}}^* \neq \epsilon_{\text{vap}}^*$

TAB $\{\tau_i, P_i^{\text{sat,e}}\}_i \rightsquigarrow \tau \mapsto P^{\text{sat}}$
 $\{(\tau_i, \epsilon_i), (P_\alpha^e)_i\}_i \rightsquigarrow (\tau, \epsilon) \mapsto P_\alpha$