NUMERICAL SIMULATION WITH FINITE VOLUME OF DYNAMIC LIQUID-VAPOR PHASE TRANSITION

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BOILING CRISIS

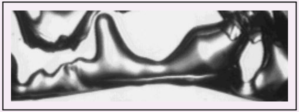
PHENOMENON

Liquid phase heated by a wall at a fixed temperature T^{wall} (pool boiling).
When T^{wall} increases, we switch from a nucleate boiling to a film boiling.

Nucleate Boiling

→

Film Boiling



source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm

OUTLINE

OUTLINE

EULER SYSTEM

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u} + P\,\mathbb{I}) = \mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}}, \\ \partial_t\left(\rho\left(\frac{|\mathbf{u}|^2}{2}+\varepsilon\right)\right) + \operatorname{div}\left(\rho\left(\frac{|\mathbf{u}|^2}{2}+\varepsilon\right)\mathbf{u} + P\,\mathbf{u}\right) = (\mathfrak{V}_{\text{vf}} - \mathfrak{S}_{\text{sf}})\cdot\mathbf{u} - \operatorname{div}(q). \end{cases}$$

- $(\mathbf{x},t)\mapsto\rho$ specific density,
- $(\mathbf{x},t)\mapsto\varepsilon$ specific internal energy,
- $(\mathbf{x},t)\mapsto\mathbf{u}$ velocity;
- $(\rho,\varepsilon)\mapsto\mathfrak{V}_{\text{vf}}$ volumic forces,
- $(\rho,\varepsilon)\mapsto\mathfrak{S}_{\text{sf}}$ surface forces,
- $(\rho,\varepsilon)\mapsto\operatorname{div}(q)$ heat transfert.

$\boxed{(\rho,\varepsilon)\mapsto P}$ pressure law.

EULER SYSTEM

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}+\boxed{P}\,\mathbb{I})=\mathfrak{V}_{\text{vf}}-\mathfrak{S}_{\text{sf}},\\ \partial_t\left(\rho\left(\frac{|\mathbf{u}|^2}{2}+\varepsilon\right)\right)+\operatorname{div}\left(\rho\left(\frac{|\mathbf{u}|^2}{2}+\varepsilon\right)\mathbf{u}+\boxed{P}\,\mathbf{u}\right)=(\mathfrak{V}_{\text{vf}}-\mathfrak{S}_{\text{sf}})\cdot\mathbf{u}-\operatorname{div}(q).\end{cases}$$

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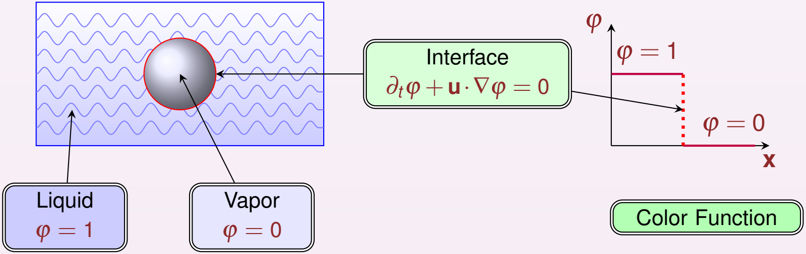
LIQUID-VAPOR INTERFACE



$$(\rho, \varepsilon) \mapsto P = \begin{cases} P^{\text{liq}} & \text{if } \varphi = 1; \\ P^{\text{vap}} & \text{if } \varphi = 0. \end{cases}$$

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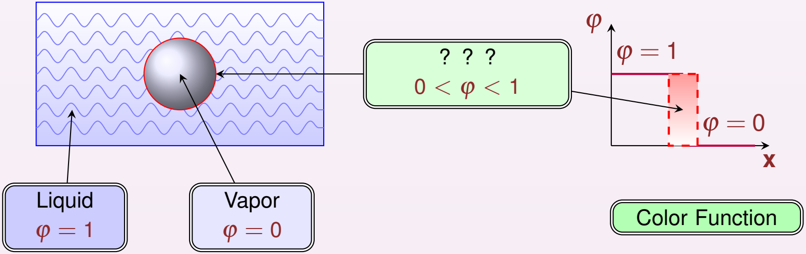

? ? ?
$0 < \varphi < 1$

Liquid
$\varphi = 1$

Vapor
$\varphi = 0$

Color Function

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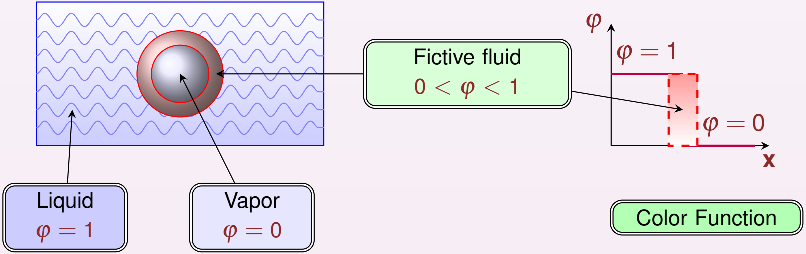

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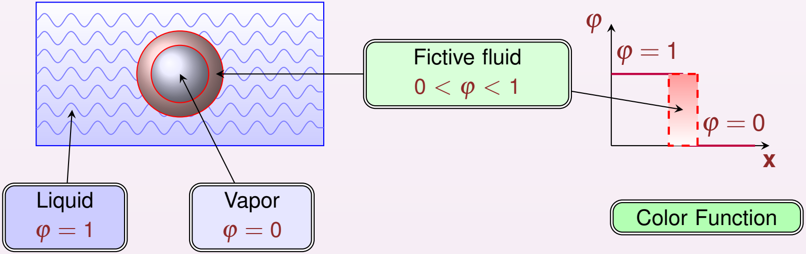

Fictive fluid
$0 < \varphi < 1$

Liquid
$\varphi = 1$

Vapor
$\varphi = 0$

Color Function

Goal: define a global pressure law such that

- $(\rho, \varepsilon, \mathbf{u}, P)$ are continuous (3 zones)
- the interface position and the phase change are implicit ($\rightsquigarrow \cancel{\varphi}$)
- coherence with classical thermodynamics [H. Callen]

EOS OF EACH PHASE $\alpha = 1,2$

$$\left.\begin{array}{l}\tau_\alpha \text{ specific volume}\\ \varepsilon_\alpha \text{ specific internal energy}\end{array}\right\} \Rightarrow \quad \mathbf{w}_\alpha \overset{\text{def}}{=} (\tau_\alpha, \varepsilon_\alpha);$$

$$\boxed{\mathbf{w}_\alpha \mapsto s_\alpha \text{ specific entropy (Hessian matrix neg. def.);}}$$

$$\begin{cases} T_\alpha \overset{\text{def}}{=} \left(\left.\dfrac{\partial s_\alpha}{\partial \varepsilon_\alpha}\right|_{\tau_\alpha} \right)^{-1} > 0 & \text{temperature,}\\ P_\alpha \overset{\text{def}}{=} T_\alpha \left.\dfrac{\partial s_\alpha}{\partial \tau_\alpha}\right|_{\varepsilon_\alpha} > 0 & \text{pressure,}\\ g_\alpha \overset{\text{def}}{=} \varepsilon_\alpha + P_\alpha \tau_\alpha - T_\alpha s_\alpha & \text{free enthalpy (Gibbs potential).} \end{cases}$$

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- $\mathbf{w} \overset{\text{def}}{=} y\mathbf{w}_1 + (1-y)\mathbf{w}_2$;
- y mass fraction;
- z volume fraction s.t. $y\tau_1 = z\tau$;
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ENTROPY WITHOUT PHASE CHANGE

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ENTROPY WITHOUT PH.CH.

$(\mathbf{w}, z, y, \psi) \longmapsto \sigma$

ENTROPY AT EQUILIBRIUM

$\mathbf{w} \longmapsto s^{\text{eq}}$

DEFINITION [H. CALLEN, PH. HELLUY . . .]

Optimization Problem:

$$s^{\text{eq}}(\mathbf{w}) \stackrel{\text{def}}{=} \max_{z,y,\psi \in [0,1]^3} \sigma(\mathbf{w}, z, y, \psi)$$

Optimality Condition: $\begin{cases} T_1(z,y,\psi) = T_2(z,y,\psi) \\ P_1(z,y,\psi) = P_2(z,y,\psi) \\ g_1(z,y,\psi) = g_2(z,y,\psi) \\ z, y, \psi \in \,]0,1[^3 \end{cases}$

Solution: (z^*, y^*, ψ^*)

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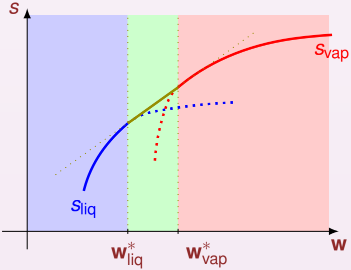

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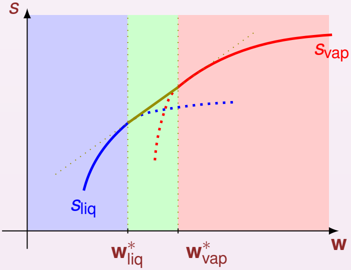

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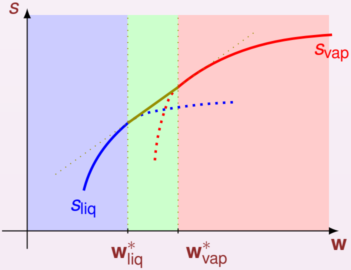

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FROM $\mathbf{w} \longmapsto s^{\mathrm{eq}}$ TO $\mathbf{w} \longmapsto P^{\mathrm{eq}}$

For all $\widetilde{\mathbf{w}}$ fixed, we seek $(\mathbf{w}^*_{\mathrm{liq}}, \mathbf{w}^*_{\mathrm{vap}}, y^*)$ as the solution of the system

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① if $y^* \in \,]0,1[$ then $\widetilde{\mathbf{w}}$ is an equilibrium mixture state

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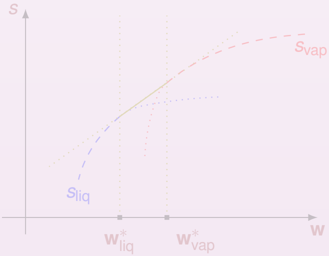

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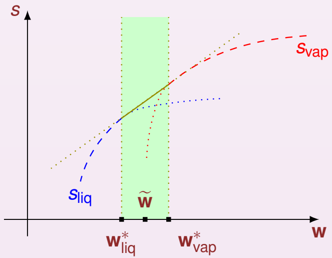

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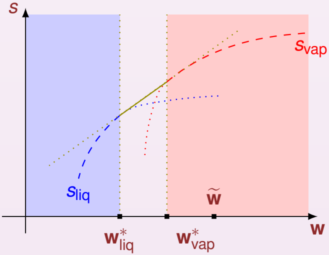

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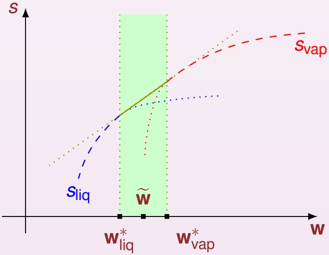

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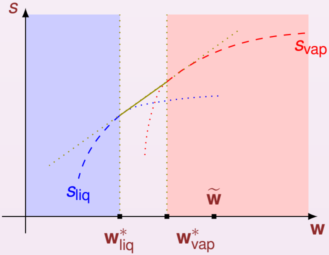

DYNAMIC LIQUID-VAPOR PHASE CHANGE

EULER SYSTEM

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + P^{\text{eq}}\, \mathbb{I}) = 0 \\ \partial_t \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \right) + \operatorname{div} \left(\rho \left(\frac{|\mathbf{u}|^2}{2} + \varepsilon \right) \mathbf{u} + P^{\text{eq}} \mathbf{u} \right) = 0 \end{cases} \quad \text{with} \quad P^{\text{eq}} \overset{\text{def}}{=} \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}.$$

PROPERTIES [G. ALLAIRE, G. FACCANONI, S. KOKH]

If $\tau_1^* \neq \tau_2^*$ and $\varepsilon_1^* \neq \varepsilon_2^*$ (first order phase transition) then

❶ $c(\mathbf{w}) > 0$, ❷ $s_{\tau\varepsilon}^{\text{eq}}(\mathbf{w}) > 0$

- ❶ Euler system: strict hyperbolicity ($\neq$ p-system),
- ❷ Riemann problem: multitude of entropic (Lax) solutions [R. Menikoff, B. J. Plohr], uniqueness of Liu solution.

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OUTLINE

1. Model
2. Numerical Method
3. Numerical Tests
4. Conclusion

HOW TO SIMULATE THE LIU SOLUTION

- Exact Riemann Solver (cf. [A. Voß] for Van der Waals EOS)
- Viscous Solver (the Liu solution is the only solution that has a viscous profile) (cf. [S. Jaouen] for Perfect Gas EOS with $c_{v_{\mathrm{liq}}} = c_{v_{\mathrm{vap}}}$)
- Solver(s) based on Relaxation Approach [F. Coquel, B. Perthame], [Th. Barberon, Ph. Helluy], [Ph. Helluy, N. Seguin], [F. Coquel, F. Caro, D. Jamet, S. Kokh],…

RELAXATION APPROACH

$$\partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

RELAXATION APPROACH

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \frac{1}{\mu}\mathbf{R}(\mathbf{V}) \qquad \xrightarrow[\mu\to 0]{\text{Formally}} \qquad \partial_t \mathbf{U} + \operatorname{div} \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

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$$P^{\text{eq}}(\rho, \varepsilon) = \frac{s_\tau^{\text{eq}}}{s_\varepsilon^{\text{eq}}}, \qquad e \overset{\text{def}}{=} \frac{|\mathbf{u}|^2}{2} + \varepsilon$$

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$$P(\rho,\varepsilon,z,y,\psi) = \frac{\sigma_\tau}{\sigma_\varepsilon}$$

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NUMERICAL SCHEME

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) \quad + \frac{1}{\mu} \quad \mathbf{R}(\mathbf{V})$$

$$\mathbf{V}_i^n \longrightarrow \mathbf{V}_i^{n+1}$$

❶ $\mu_j = +\infty$

$$\Downarrow$$

$$\partial_t \mathbf{V} + \operatorname{div} \mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V})$$

Aug. System: 5-eq. iso-T
Num. Scheme: op. splitting
Conv.: [G. Allaire and all.]
Surf. Tens.: [J. U. Brackbill and all.]
Heat: 2D implicit

❷ $\mu_j = 0$

$$\Downarrow$$

$$\mathbf{R}(\mathbf{V}) = \mathbf{0}$$

update fractions
(y, z, ψ) by
projecting $\mathbf{V}_i^{n+1/2}$
onto the
P, T, g equilibrium

NUMERICAL SCHEME

$$\partial_t \mathbf{V} + \operatorname{div}\mathbf{G}(\mathbf{V}) = \mathbf{S}(\mathbf{V}) + \frac{1}{\mu}\mathbf{R}(\mathbf{V})$$

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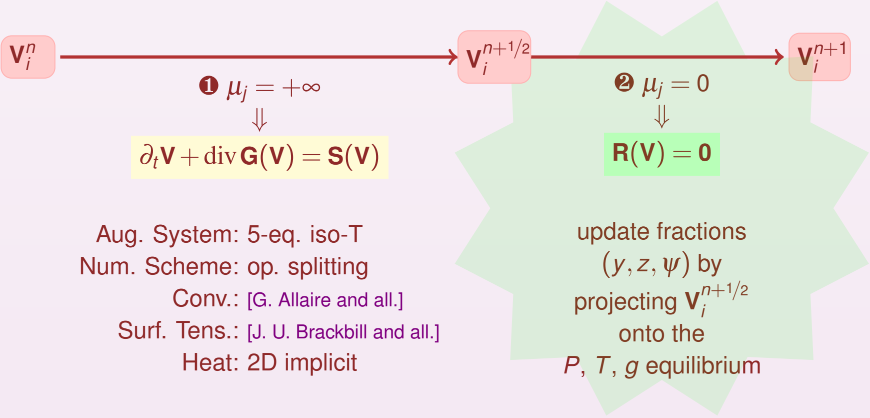

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ANALYTICAL EOS

(τ, ε) fixed

$(\tau_1, \varepsilon_1, \tau_2, \varepsilon_2, y)$ SOLUTION OF

$$\begin{cases} g_1(\tau_1, \varepsilon_1) = g_2(\tau_2, \varepsilon_2) \\ P_1(\tau_1, \varepsilon_1) = P_2(\tau_2, \varepsilon_2) \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) \\ \tau = y\tau_1 + (1-y)\tau_2 \\ \varepsilon = y\varepsilon_1 + (1-y)\varepsilon_2 \end{cases}$$

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$$\begin{cases} g_1(P, T) = g_2(P, T) \\ \dfrac{\tau - \tau_2(P, T)}{\tau_1(P, T) - \tau_2(P, T)} = \dfrac{\varepsilon - \varepsilon_2(P, T)}{\varepsilon_1(P, T) - \varepsilon_2(P, T)} \end{cases}$$

$$T \mapsto P = P^{\text{sat}}(T) \approx \mathcal{P}^{\text{sat}}(T)$$

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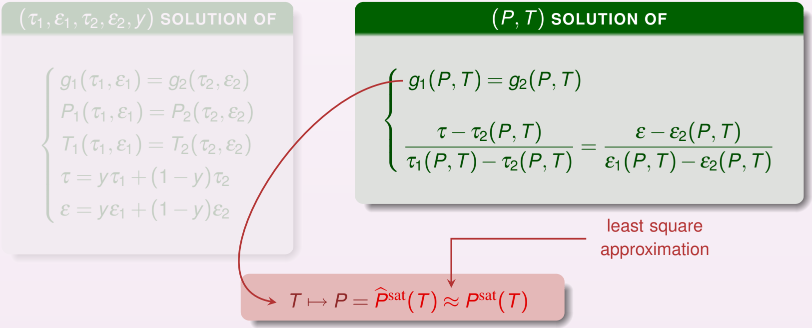

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least square approximation

$$T \mapsto P = \widehat{P}^{\text{sat}}(T) \approx P^{\text{sat}}(T)$$

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TABULATED EOS

		Volume (m^3/kg)		Internal Energy (kJ/kg)	
T (K)	P^{sat} (MPa)	$\tau_{\text{liq}}^{\text{sat}}$	$\tau_{\text{vap}}^{\text{sat}}$	$\varepsilon_{\text{liq}}^{\text{sat}}$	$\varepsilon_{\text{vap}}^{\text{sat}}$
275	0,00069845	0,0010001	181,60	7,7590	2377,5
278	0,00086349	0,0010001	148,48	20,388	2381,6
281	0,0010621	0,0010002	122,01	32,996	2385,7
284	0,0012999	0,0010004	100,74	45,586	2389,8
287	0,0015835	0,0010008	83,560	58,162	2393,9
290	0,0019200	0,0010012	69,625	70,727	2398,0
293	0,0023177	0,0010018	58,267	83,284	2402,1
296	0,0027856	0,0010025	48,966	95,835	2406,2
299	0,0033342	0,0010032	41,318	108,38	2410,3
302	0,0039745	0,0010041	35,002	120,92	2414,4
305	0,0047193	0,0010050	29,764	133,46	2418,4
308	0,0055825	0,0010060	25,403	146	2422,5
...	...	...	...	...	...

Source: `http://webbook.nist.gov/chemistry/fluid/`

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$$\frac{\tau - \tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T) - \tau_2^{\text{sat}}(T)} = \frac{\varepsilon - \varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T) - \varepsilon_2^{\text{sat}}(T)}$$ with $\begin{pmatrix}\tau\\ \varepsilon\end{pmatrix}_\alpha^{\text{sat}}(T)$ tabulated

$\wr\wr$

$$\frac{\tau - \widehat{\tau}_2^{\text{sat}}(T)}{\widehat{\tau}_1^{\text{sat}}(T) - \widehat{\tau}_2^{\text{sat}}(T)} = \frac{\varepsilon - \widehat{\varepsilon}_2^{\text{sat}}(T)}{\widehat{\varepsilon}_1^{\text{sat}}(T) - \widehat{\varepsilon}_2^{\text{sat}}(T)}$$ with $\begin{pmatrix}\widehat{\tau}\\ \widehat{\varepsilon}\end{pmatrix}_\alpha^{\text{sat}}(T)$

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(τ, ε) fixed

T SOLUTION OF

$$\frac{\tau-\tau_2^{\text{sat}}(T)}{\tau_1^{\text{sat}}(T)-\tau_2^{\text{sat}}(T)}=\frac{\varepsilon-\varepsilon_2^{\text{sat}}(T)}{\varepsilon_1^{\text{sat}}(T)-\varepsilon_2^{\text{sat}}(T)}$$

with $\begin{pmatrix}\tau\\ \varepsilon\end{pmatrix}_\alpha^{\text{sat}}(T)$ tabulated

$$\wr$$

$$\frac{\tau-\widehat{\tau}_2^{\text{sat}}(T)}{\widehat{\tau}_1^{\text{sat}}(T)-\widehat{\tau}_2^{\text{sat}}(T)}=\frac{\varepsilon-\widehat{\varepsilon}_2^{\text{sat}}(T)}{\widehat{\varepsilon}_1^{\text{sat}}(T)-\widehat{\varepsilon}_2^{\text{sat}}(T)}$$

with $\begin{pmatrix}\widehat{\tau}\\ \widehat{\varepsilon}\end{pmatrix}_\alpha^{\text{sat}}(T)$

least square approximations

TABULATED EOS

(τ,ε) fixed

T SOLUTION OF

$$\frac{\tau-\tau_2^{\mathrm{sat}}(T)}{\tau_1^{\mathrm{sat}}(T)-\tau_2^{\mathrm{sat}}(T)}=\frac{\varepsilon-\varepsilon_2^{\mathrm{sat}}(T)}{\varepsilon_1^{\mathrm{sat}}(T)-\varepsilon_2^{\mathrm{sat}}(T)} \quad \text{with} \quad \begin{pmatrix}\tau\\ \varepsilon\end{pmatrix}_\alpha^{\mathrm{sat}}(T) \quad \text{tabulated}$$

$$\wr\wr$$

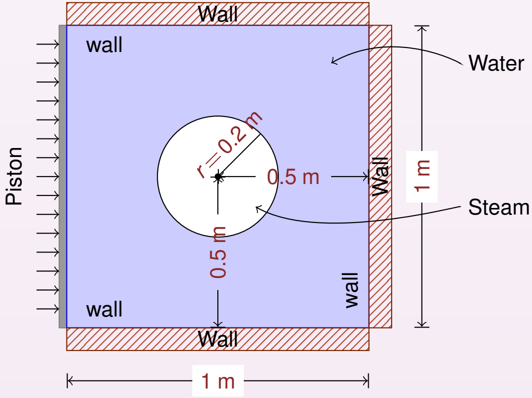
$$\frac{\tau-\widehat{\tau}_2^{\mathrm{sat}}(T)}{\widehat{\tau}_1^{\mathrm{sat}}(T)-\widehat{\tau}_2^{\mathrm{sat}}(T)}=\frac{\varepsilon-\widehat{\varepsilon}_2^{\mathrm{sat}}(T)}{\widehat{\varepsilon}_1^{\mathrm{sat}}(T)-\widehat{\varepsilon}_2^{\mathrm{sat}}(T)} \quad \text{with} \quad \begin{pmatrix}\widehat{\tau}\\ \widehat{\varepsilon}\end{pmatrix}_\alpha^{\mathrm{sat}}(T)$$

least square approximations

OUTLINE

1. Model
2. Numerical Method
3. Numerical Tests
4. Conclusion

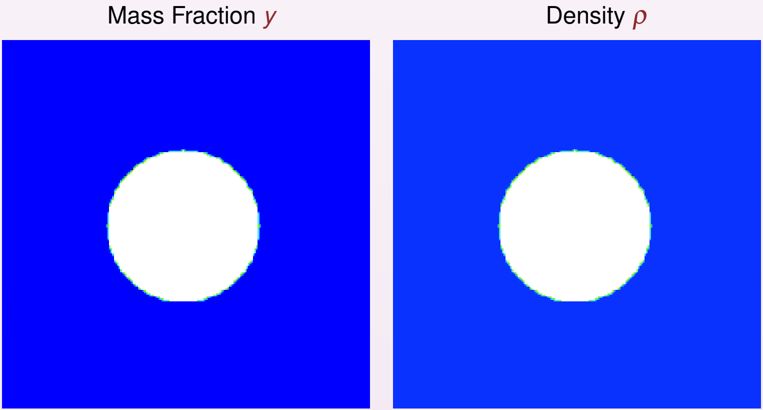
COMPRESSION OF A VAPOR BUBBLE



Compression of a 2D Vapor Bubble involving two Stiffened Gases for water and steam.

The piston moves towards right at constant speed $u_\mathrm{p} = 30$ m/s.

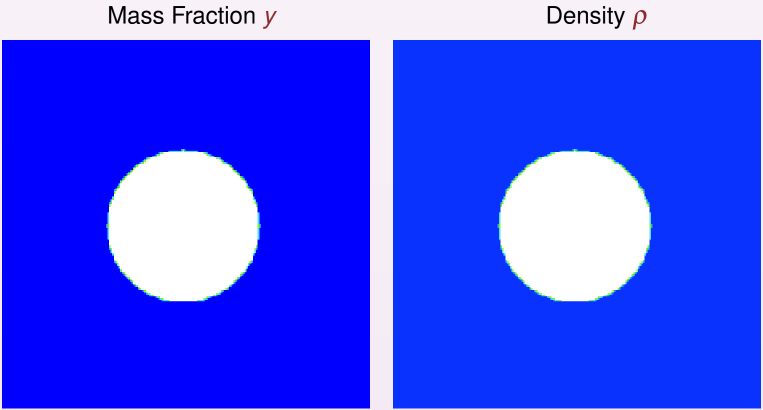
COMPRESSION OF A VAPOR BUBBLE



$t = 0.00$ ms

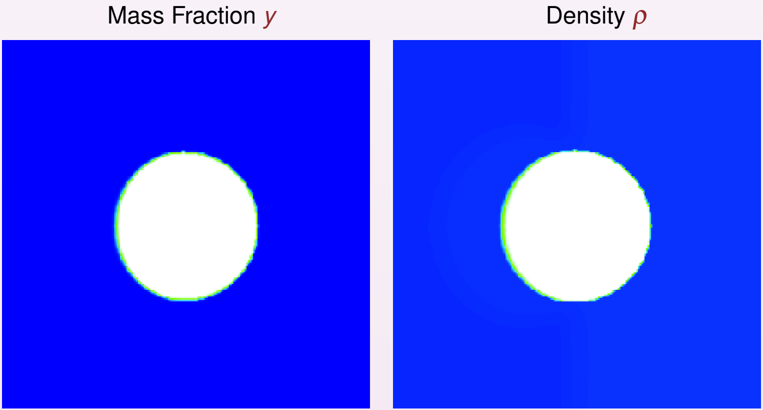
◀ Geometry ▶ Play ⏩ Skip

COMPRESSION OF A VAPOR BUBBLE



$t = 0.00$ ms

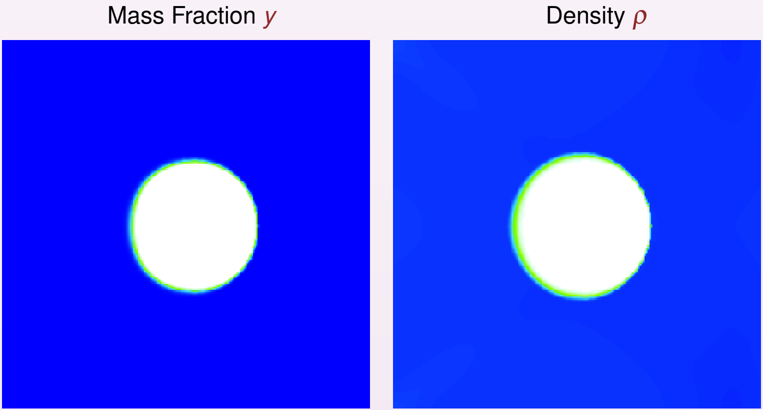
COMPRESSION OF A VAPOR BUBBLE



$t = 0.89$ ms

◂ Geometry ▸ Play ▸▸ Skip

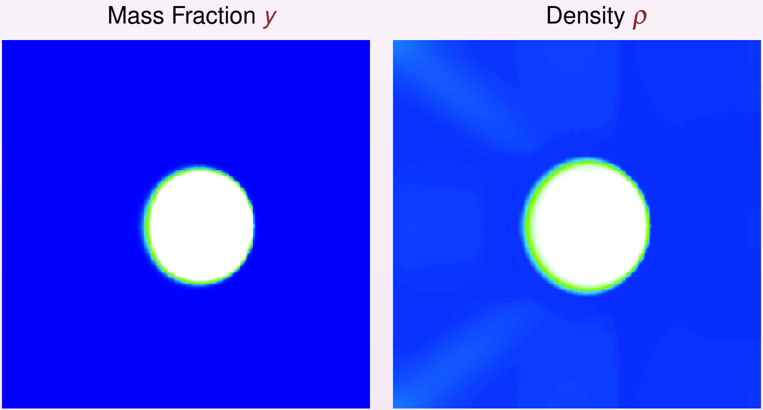
COMPRESSION OF A VAPOR BUBBLE



$t = 1.09$ ms

◂ Geometry ▸ Play ▸▸ Skip

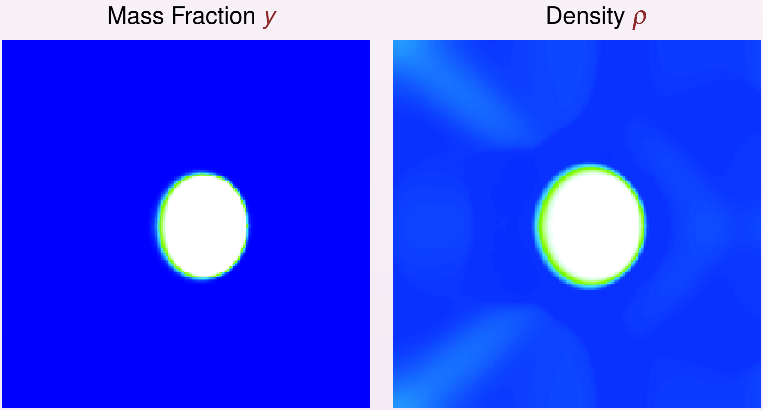
COMPRESSION OF A VAPOR BUBBLE



$t = 1.49$ ms

◂ Geometry ▸ Play ▸▸ Skip

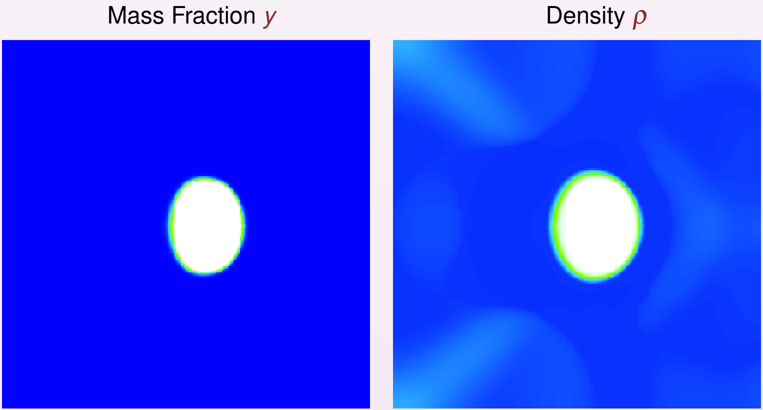
COMPRESSION OF A VAPOR BUBBLE



$t = 1.80$ ms

◂ Geometry ▸ Play ▸▸ Skip

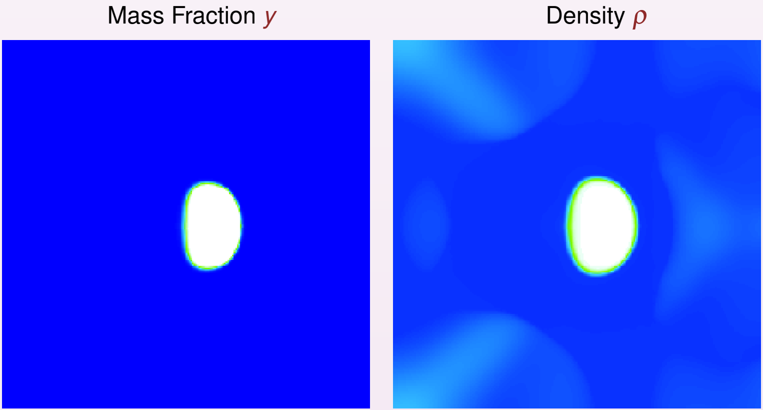
COMPRESSION OF A VAPOR BUBBLE



$t = 2.09$ ms

◂ Geometry ▸ Play ▸▸ Skip

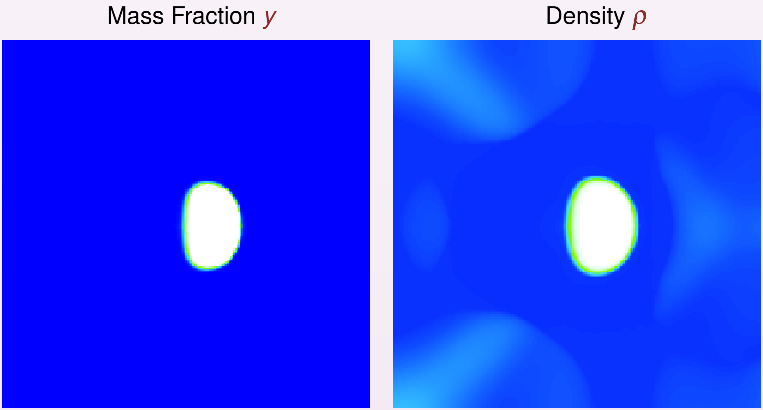
COMPRESSION OF A VAPOR BUBBLE



$t = 2.39$ ms

◂ Geometry ▸ Play ▸▸ Skip

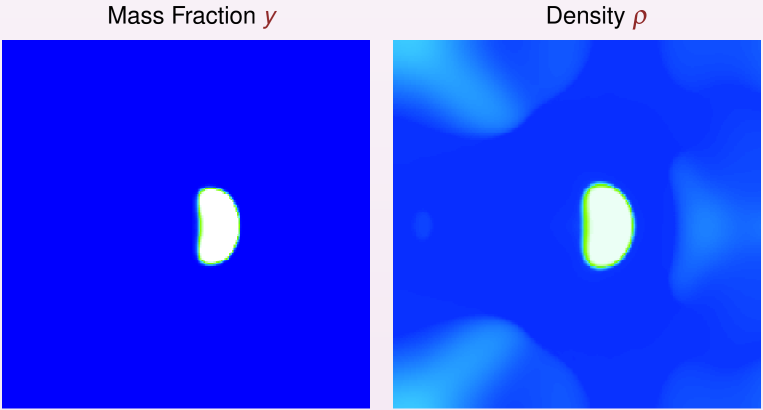
COMPRESSION OF A VAPOR BUBBLE



$t = 2.69$ ms

◂ Geometry ▸ Play ▸▸ Skip

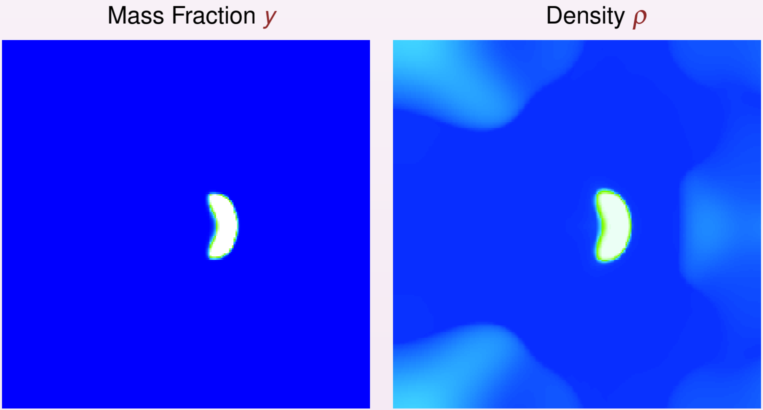
COMPRESSION OF A VAPOR BUBBLE



$t = 2.99$ ms

◂ Geometry ▸ Play ▸▸ Skip

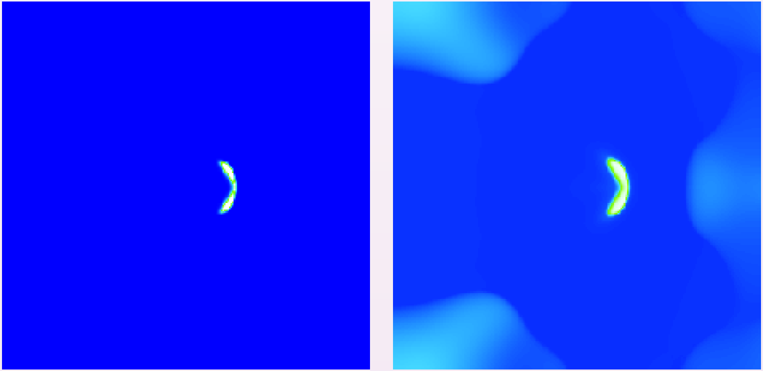
COMPRESSION OF A VAPOR BUBBLE



$t = 3.29$ ms

COMPRESSION OF A VAPOR BUBBLE

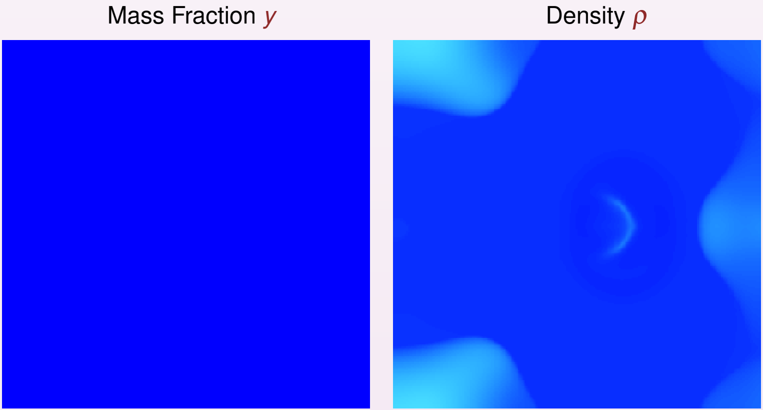
Mass Fraction y Density ρ



$t = 3.49$ ms

◂ Geometry ▸ Play ▸▸ Skip

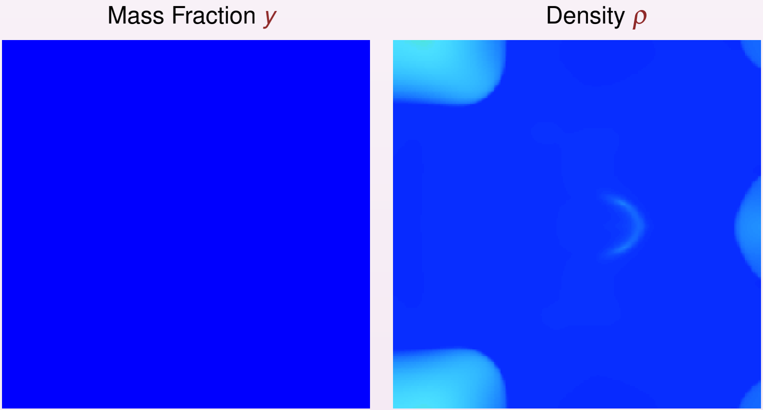
COMPRESSION OF A VAPOR BUBBLE



$t = 3.60$ ms

◂ Geometry ▸ Play ▸▸ Skip

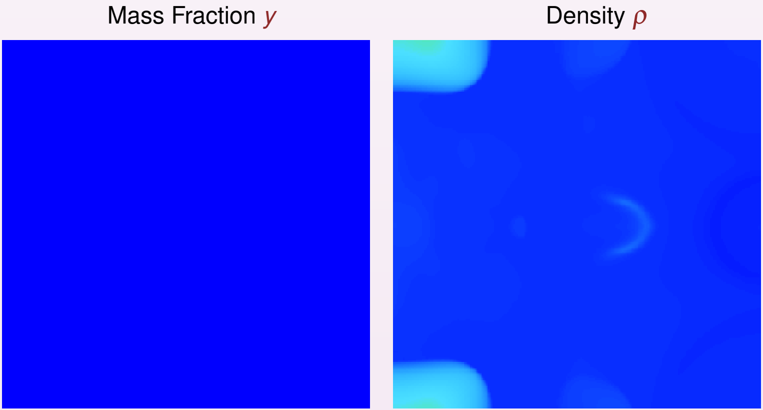
COMPRESSION OF A VAPOR BUBBLE



$t = 3.80$ ms

◂ Geometry ▸ Play ▸▸ Skip

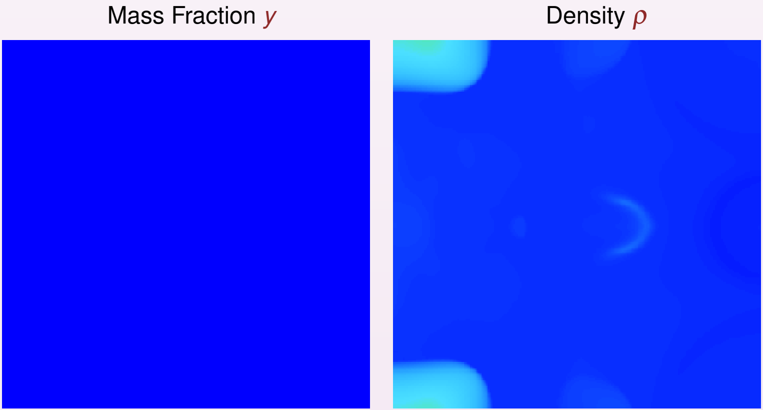
COMPRESSION OF A VAPOR BUBBLE



$t = 3.99$ ms

◀ Geometry ▶ Play ⏩ Skip

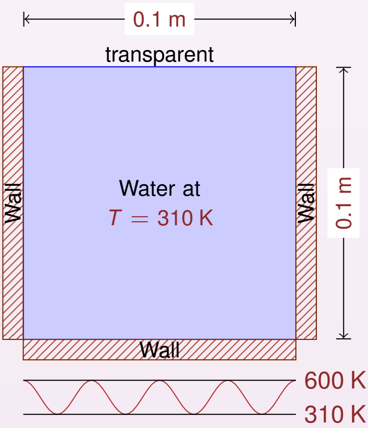
COMPRESSION OF A VAPOR BUBBLE



$t = 4.10$ ms

◀ Geometry ▶ Play ⏩ Skip

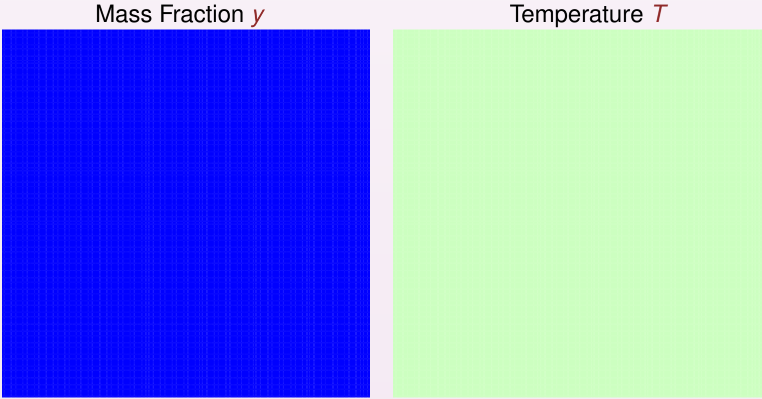
NUCLEATING BUBBLES



Nucleation of a 2D Vapor Bubbles involving two Stiffened Gases for water and steam. The temperature of the south wall is fixed at

$$T^{\text{wall}} = 310 + (600 - 310)(1 + \cos(6\pi x))/2.$$

NUCLEATING BUBBLES

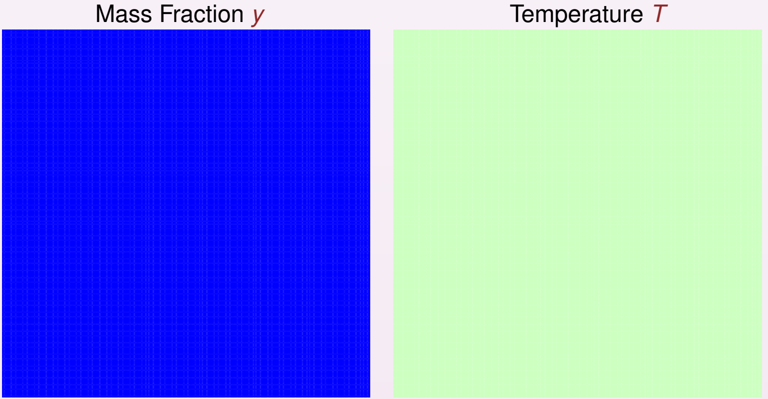


Mass Fraction y

Temperature T

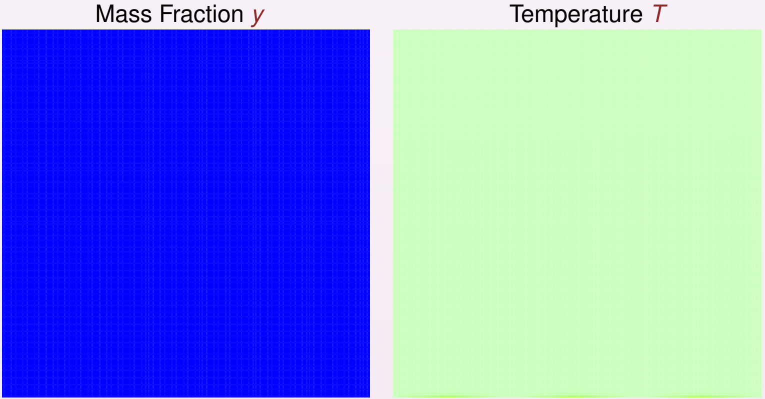
◀ Geometry ▶ Play ▶▶ Skip

NUCLEATING BUBBLES



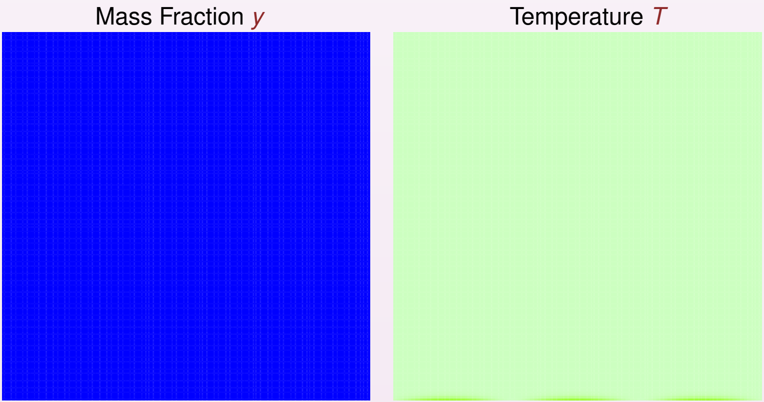
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



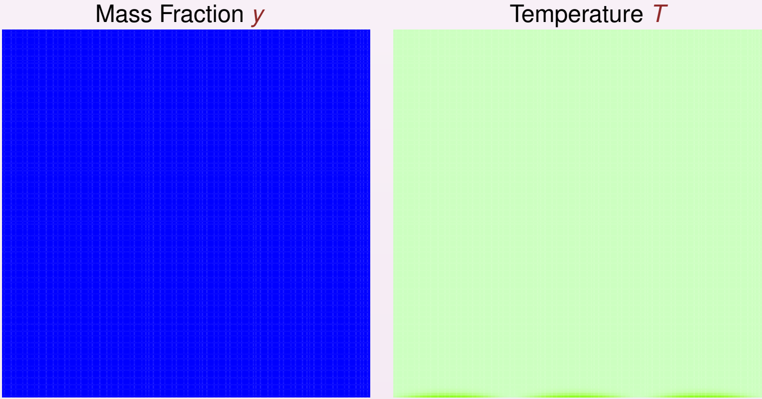
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

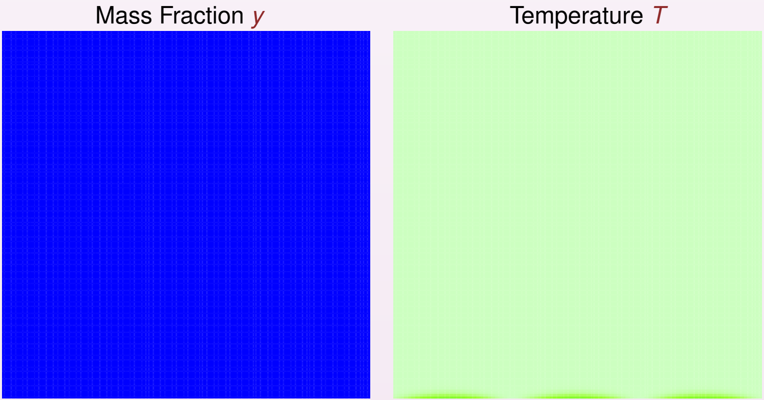


Mass Fraction y

Temperature T

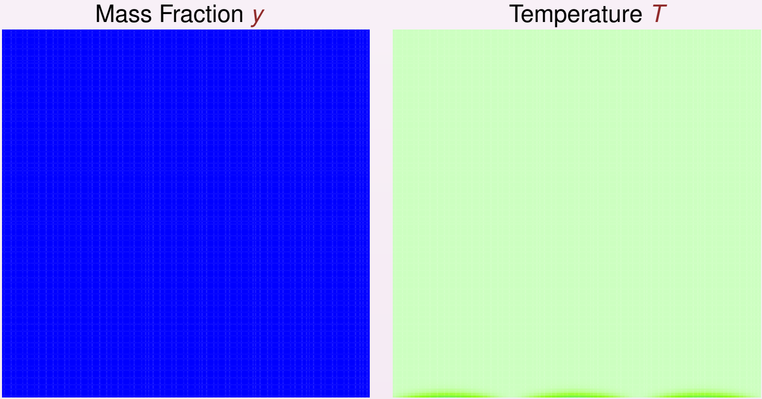
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



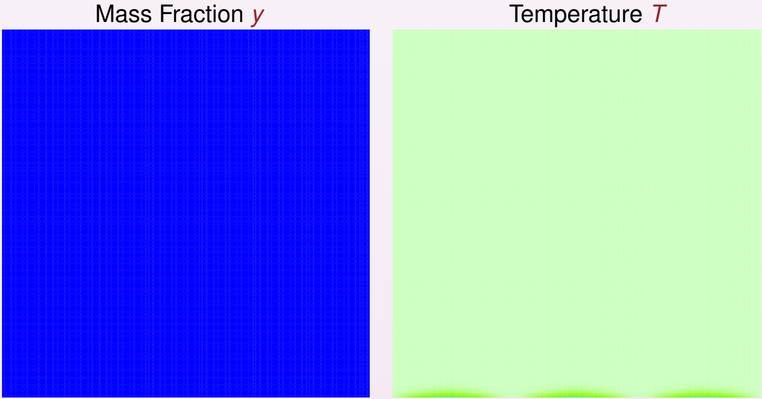
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



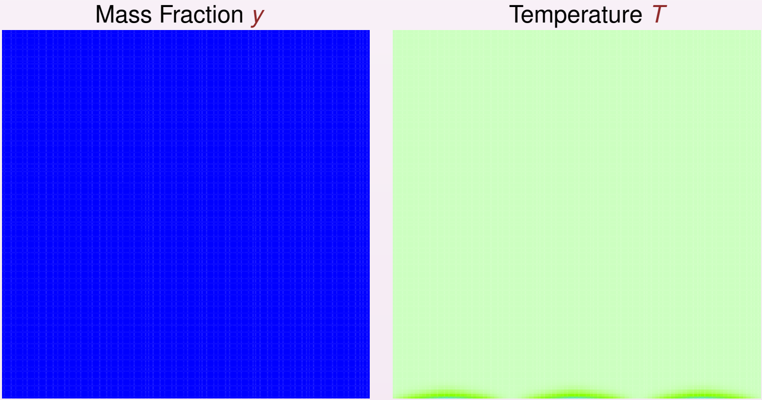
◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES



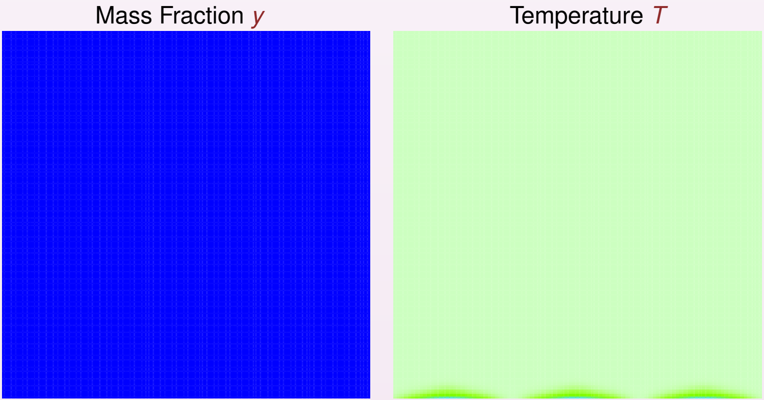
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



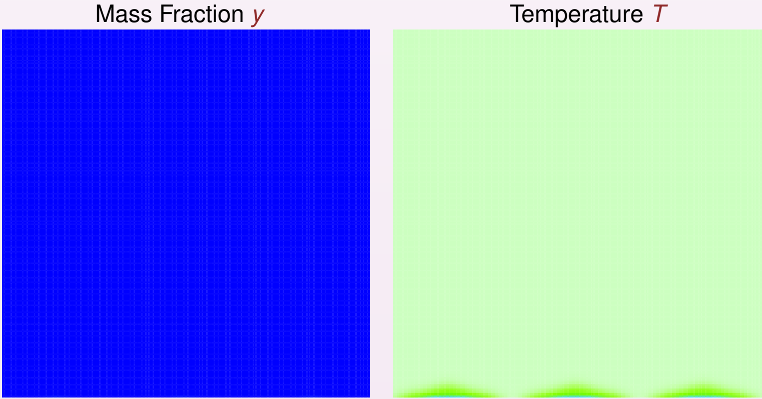
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



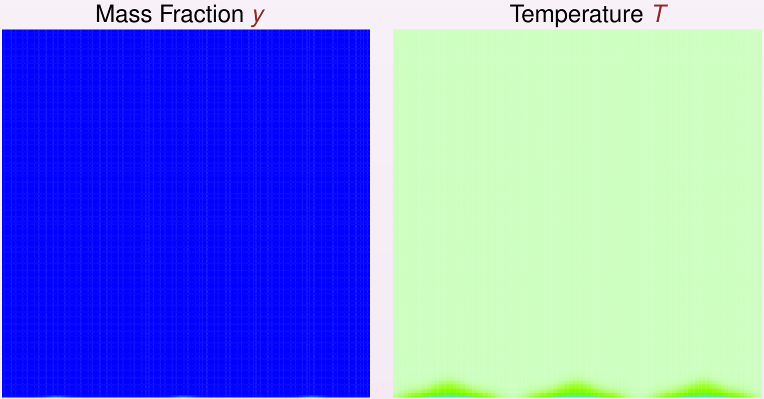
◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES



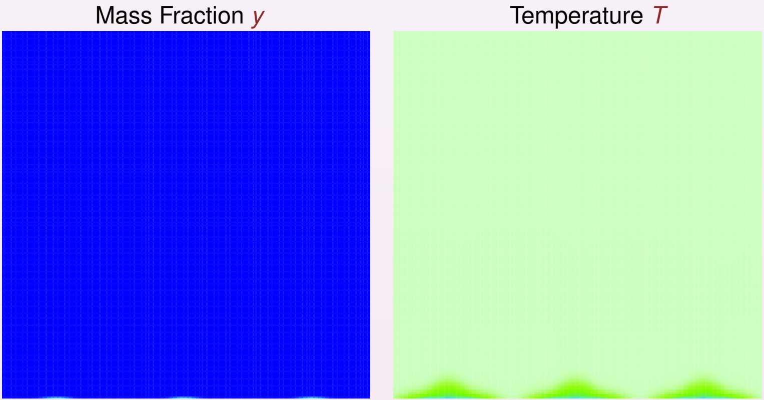
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

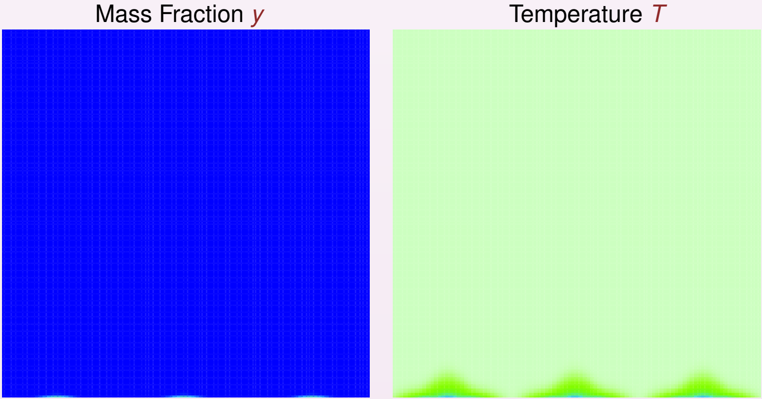


◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES



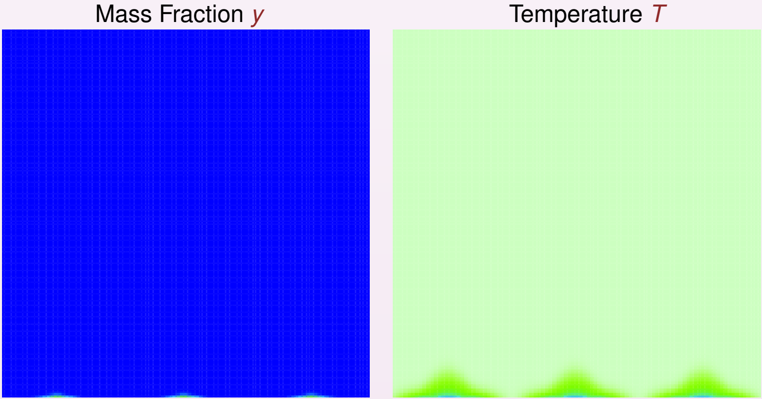
NUCLEATING BUBBLES


Mass Fraction y

Temperature T

◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

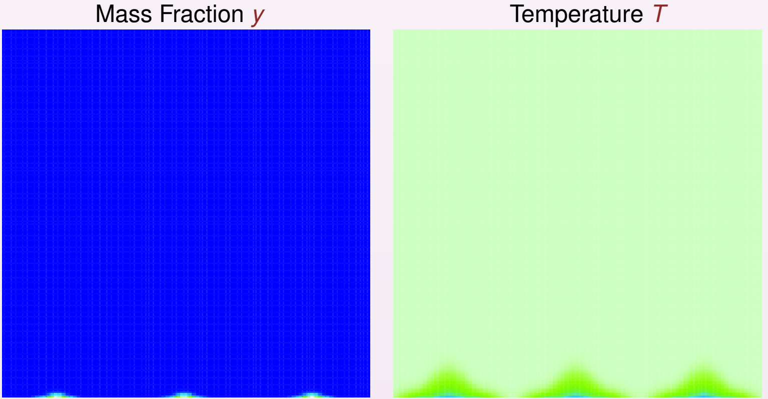


Mass Fraction y

Temperature T

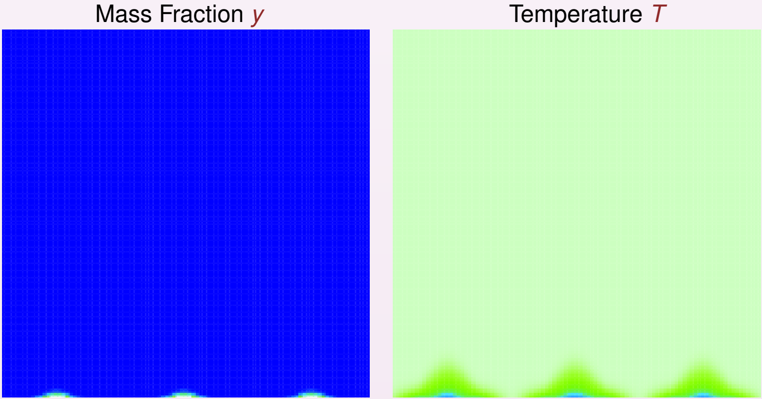
◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES



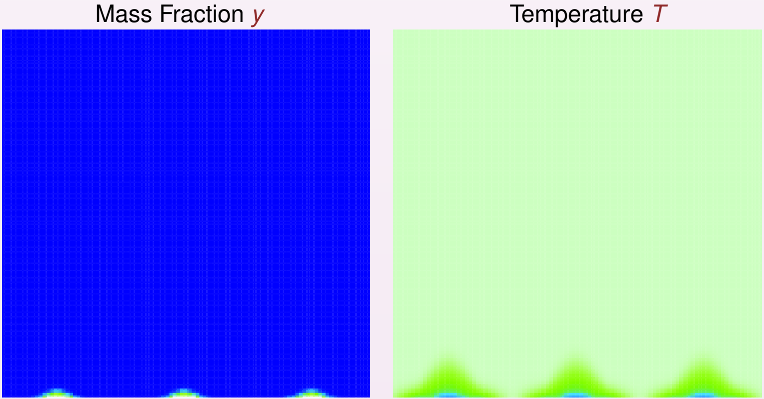
◂ Geruch

NUCLEATING BUBBLES



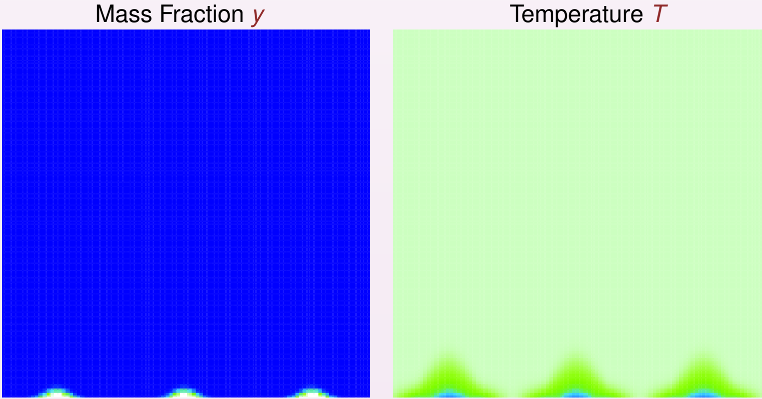
◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES



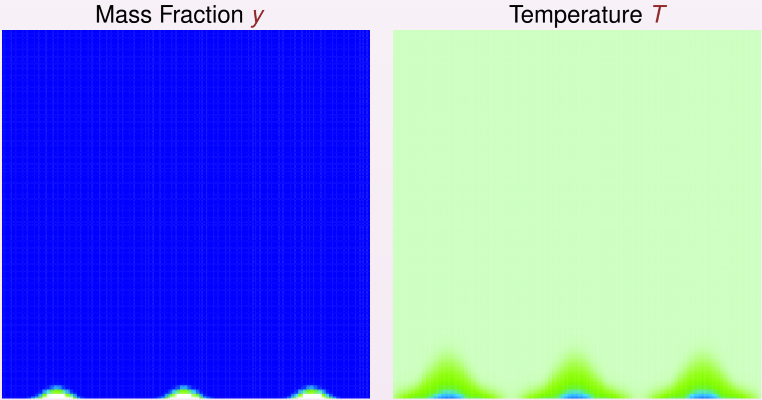
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



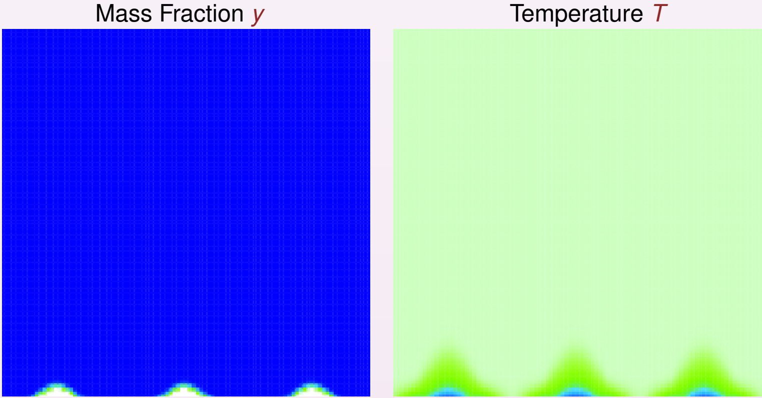
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



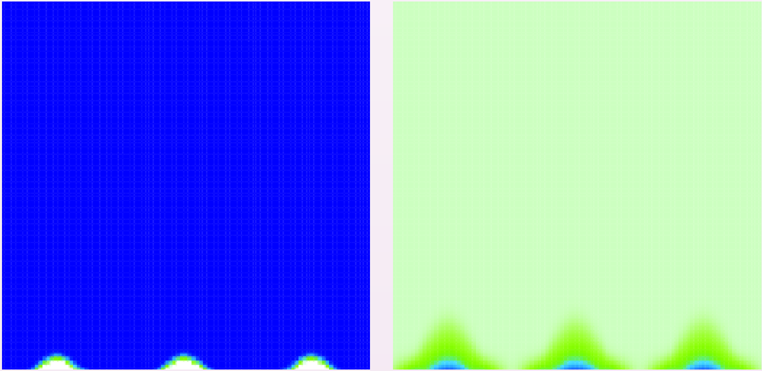
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



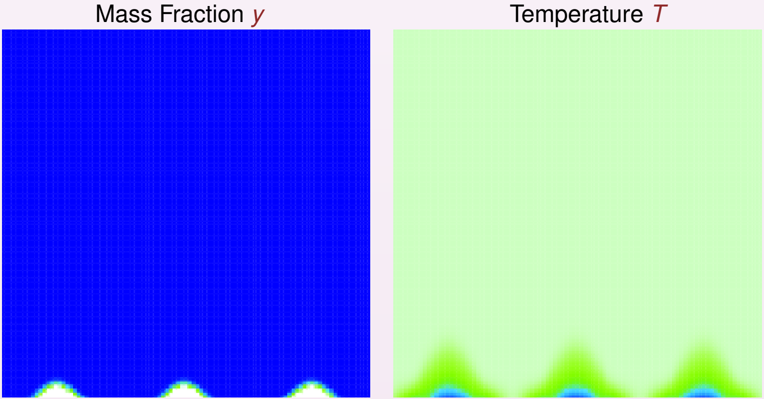
NUCLEATING BUBBLES

Mass Fraction y Temperature T



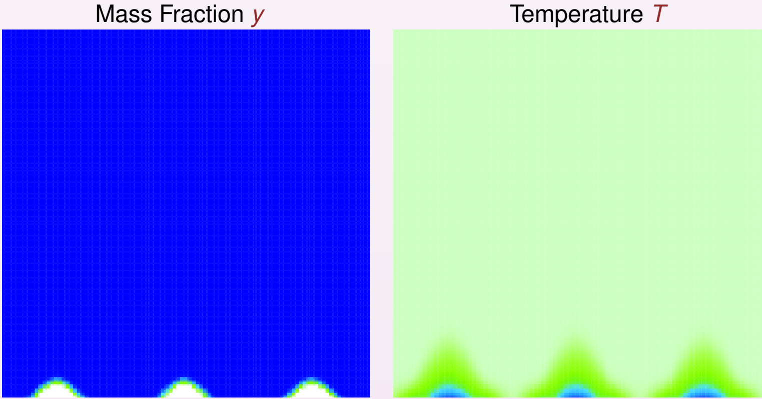
◀ Geometry ▶ Play ▶▶ Skip

NUCLEATING BUBBLES



◂ Geometry ▸ Play ▸▸ Skip

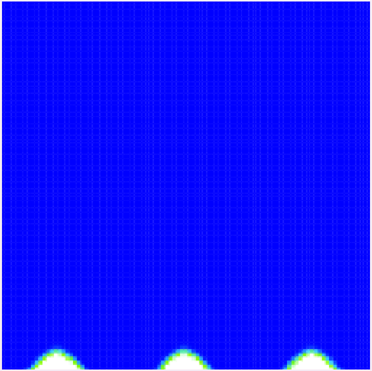
NUCLEATING BUBBLES



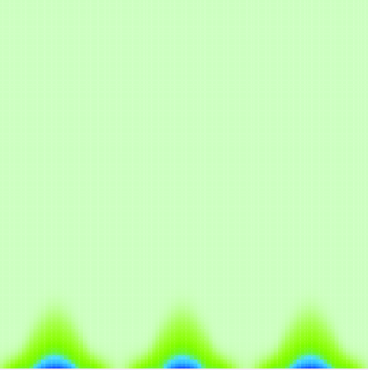
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

Mass Fraction y

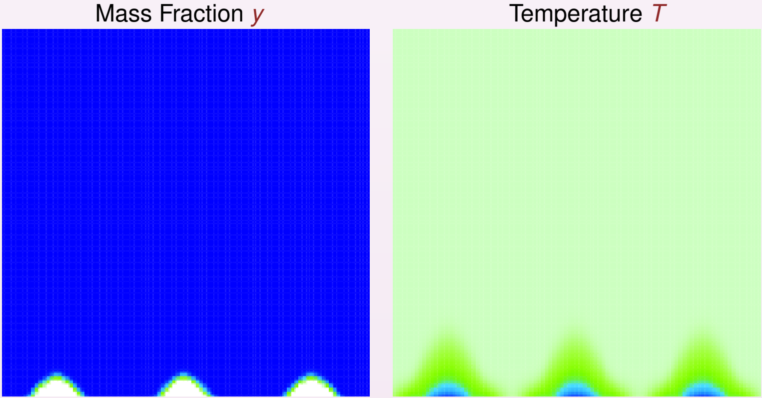


Temperature T



◂ Geometry ▸ Play ▸▸ Skip

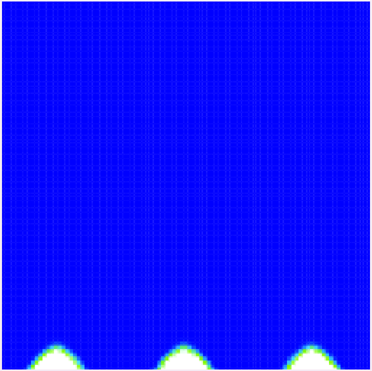
NUCLEATING BUBBLES



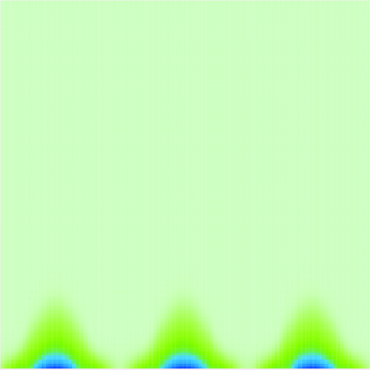
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

Mass Fraction y

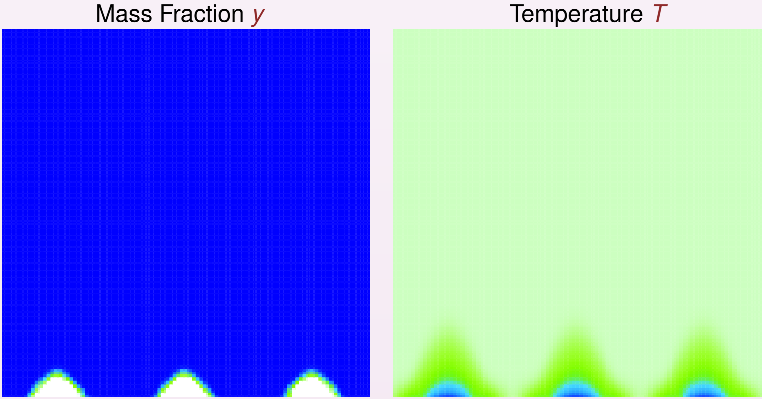


Temperature T

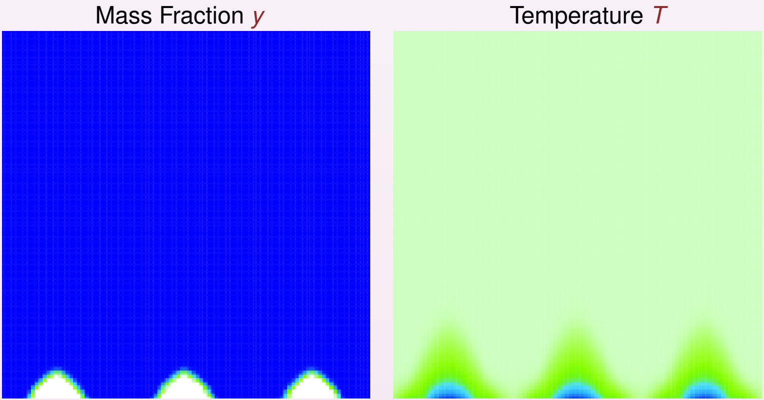


◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

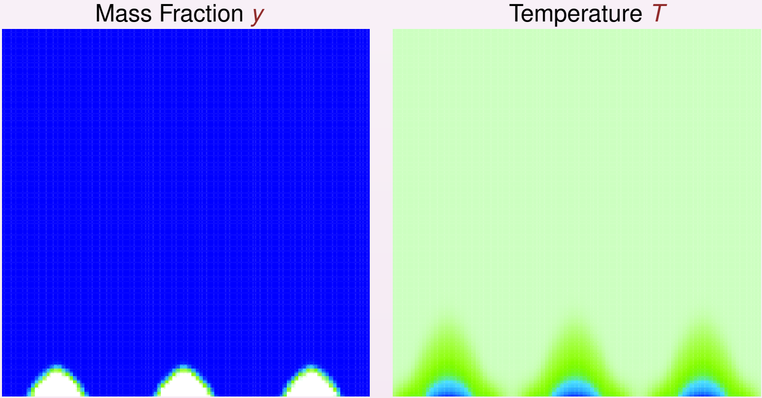


NUCLEATING BUBBLES



◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES

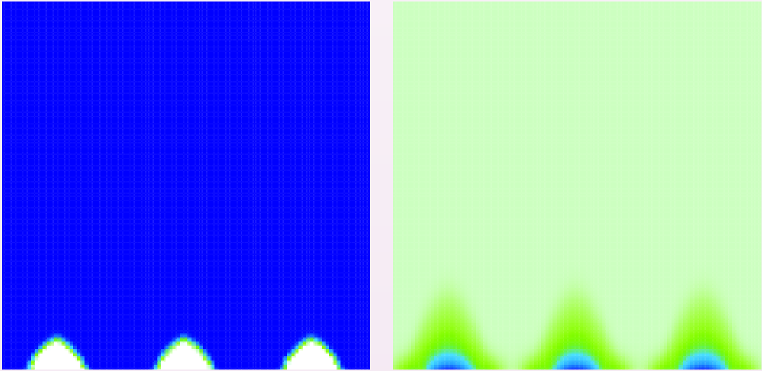


◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES

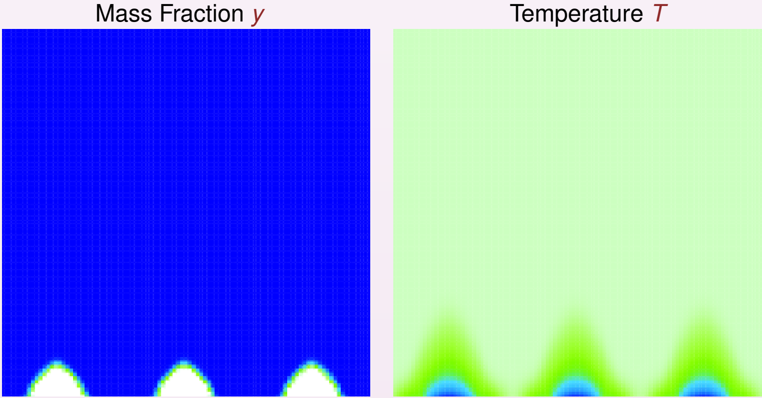
Mass Fraction y

Temperature T



◀ Geometry ▶ Play ▶▶ Skip

NUCLEATING BUBBLES

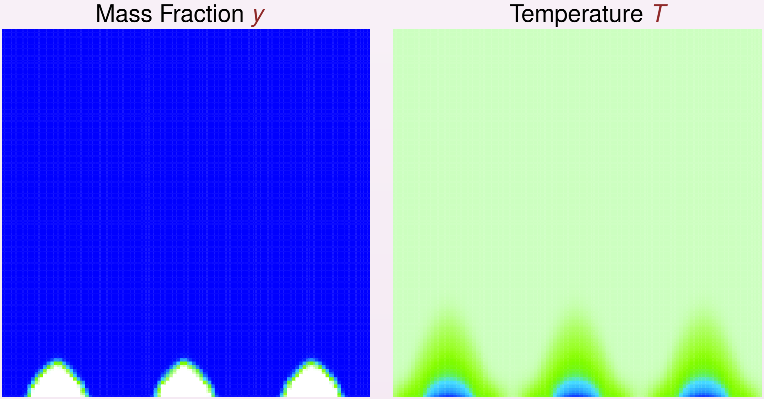


Mass Fraction y

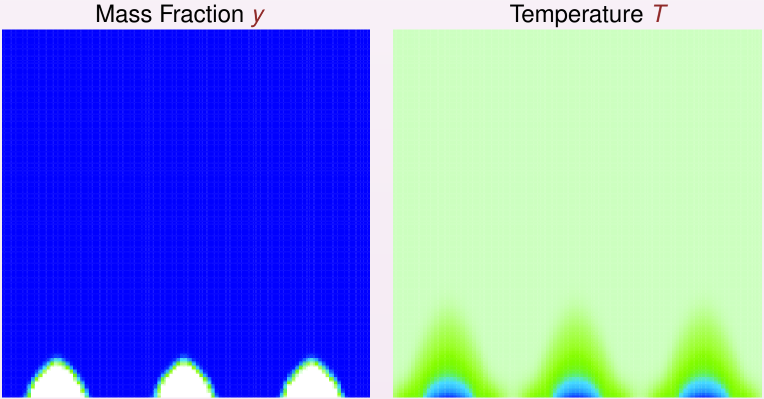
Temperature T

◂ Geometry ▸ Play ▸▸ Skip

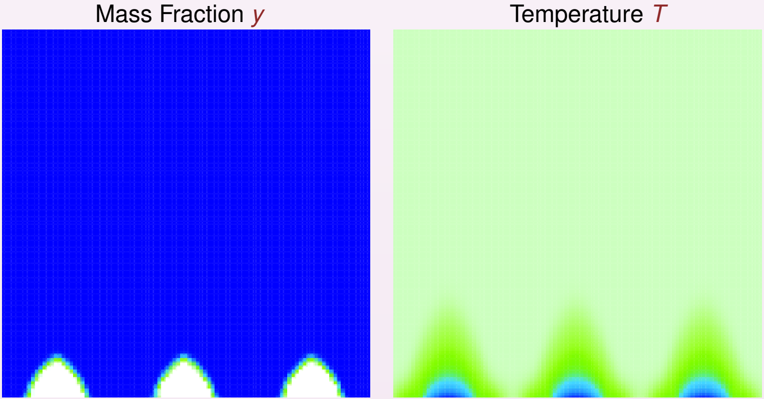
NUCLEATING BUBBLES



NUCLEATING BUBBLES

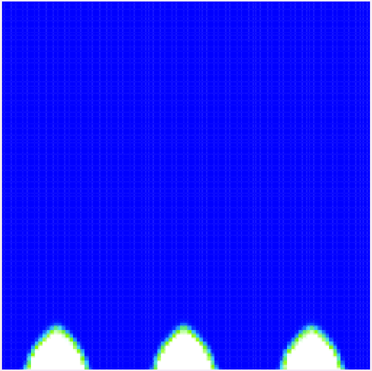


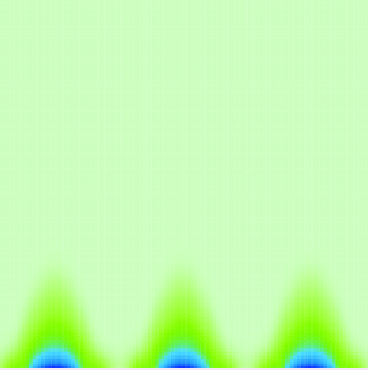
NUCLEATING BUBBLES



◂ Geometry ▸ Play ▸▸ Skip

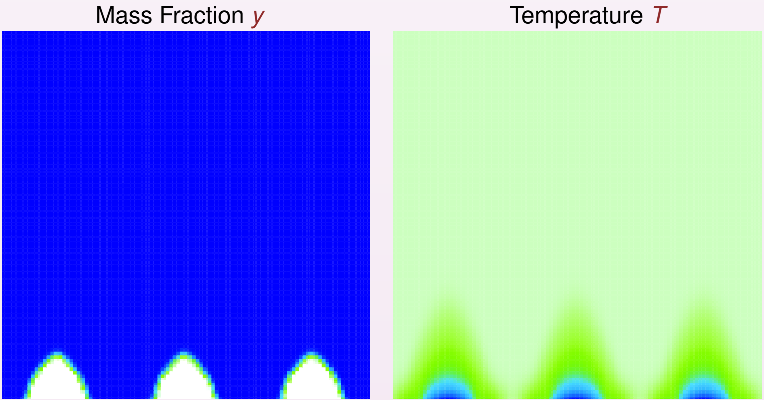
NUCLEATING BUBBLES


Mass Fraction y


Temperature T

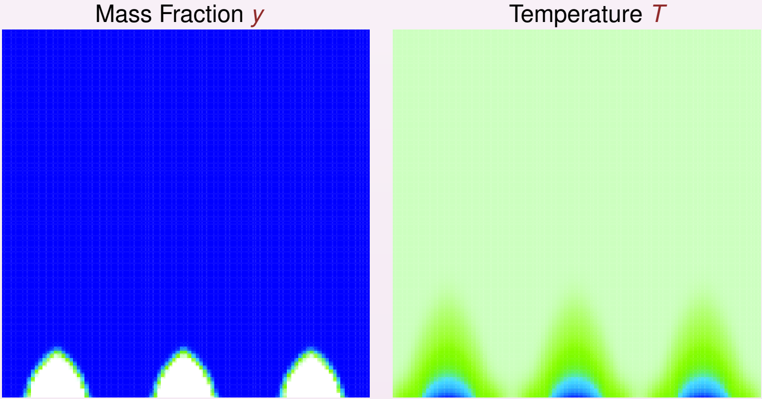
◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



◂ Geometry ▸ Play ▸▸ Skip

NUCLEATING BUBBLES



◂ Geometry ▸ Play ▸▸ Skip

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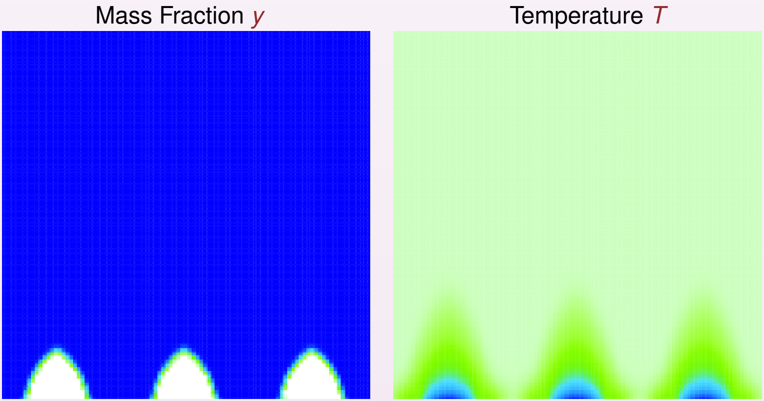
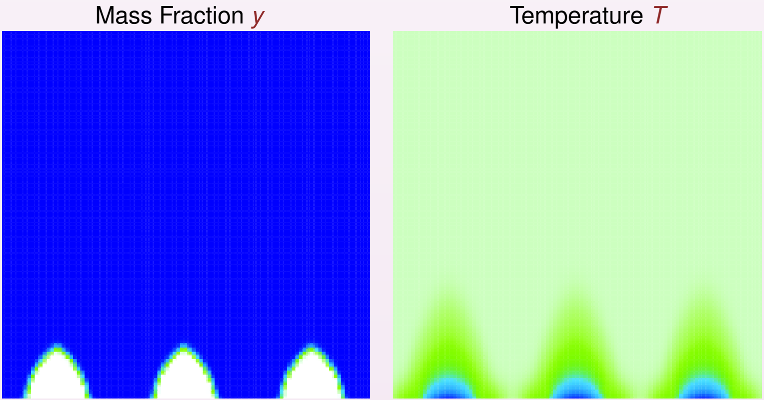


Mass Fraction y

Temperature T

◂ Geometry ▸ Play ⏩ Skip

NUCLEATING BUBBLES



◂ Geophysics

NUCLEATING BUBBLES

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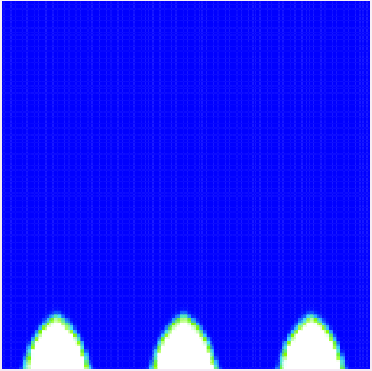
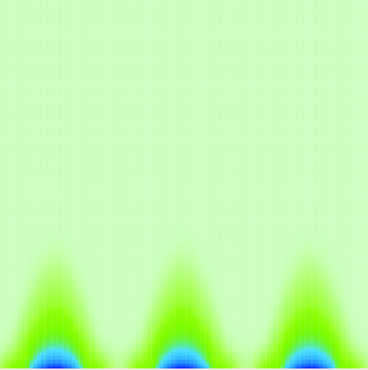


Temperature T



◀ Geometry ▶ Play ⏩ Skip

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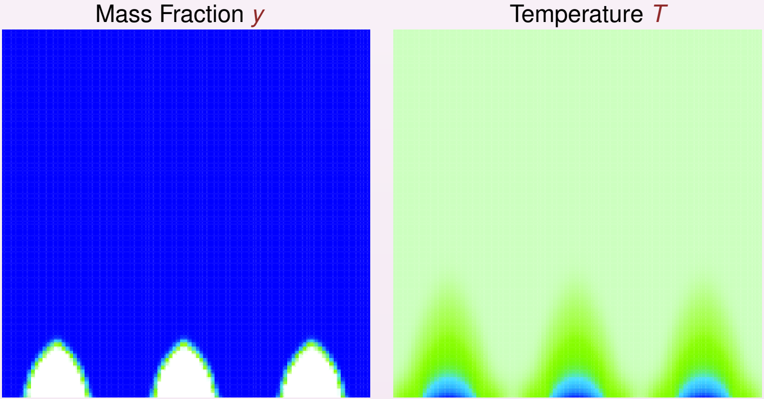

◂ Geometry ▸ Play ▸▸ Skip

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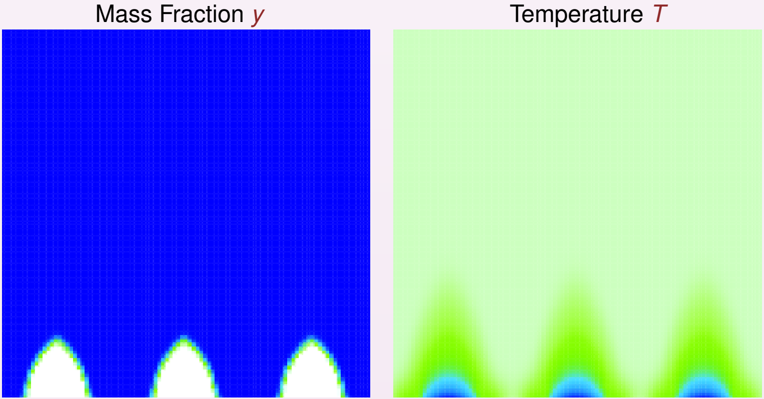

◀ Geometry ▶ Play ▶▶ Skip

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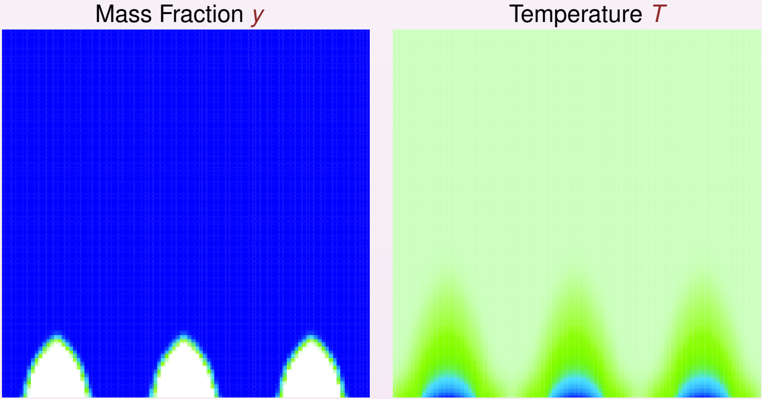

◀ Geometry ▶ Play ⏩ Skip

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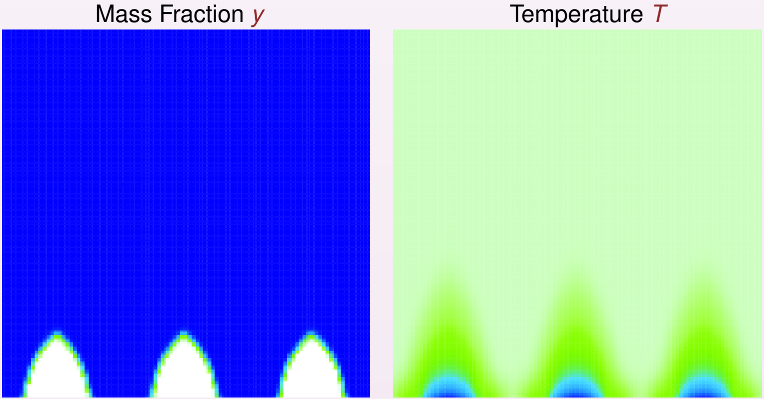

◀ Geometry ▶ Play ▶▶ Skip

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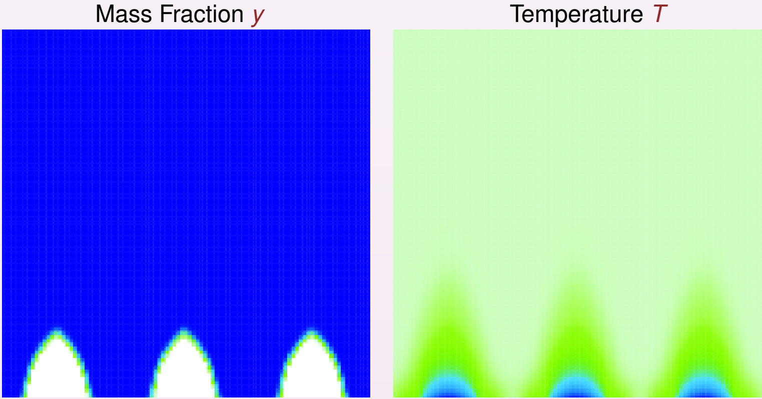

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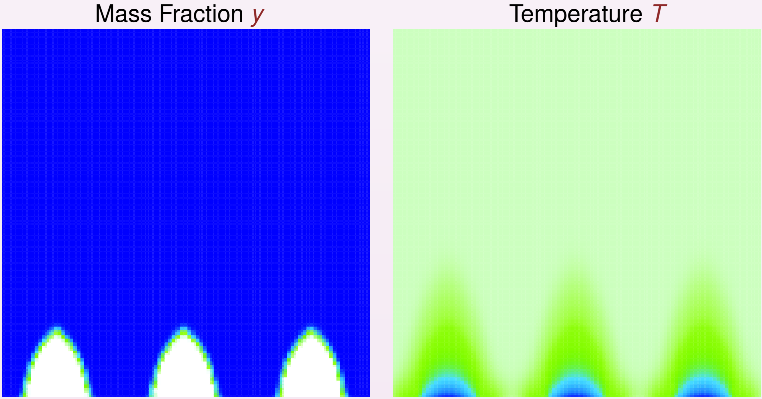

◂ Geometry ▸ Play ▸▸ Skip

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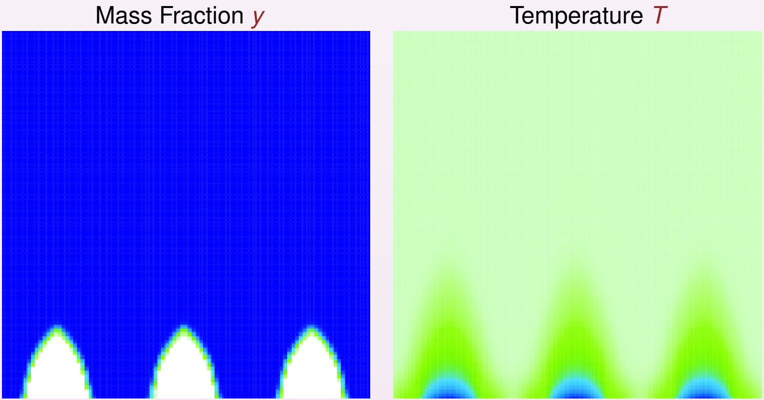

◀ Geometry ▶ Play ⏩ Skip

NUCLEATING BUBBLES

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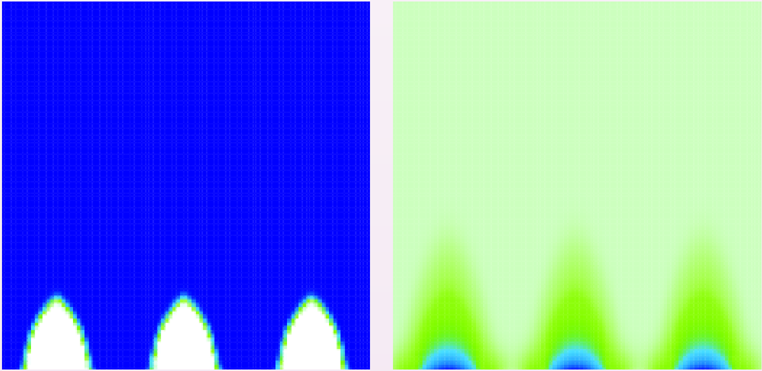

◂ Geometry ▸ Play ▸▸ Skip

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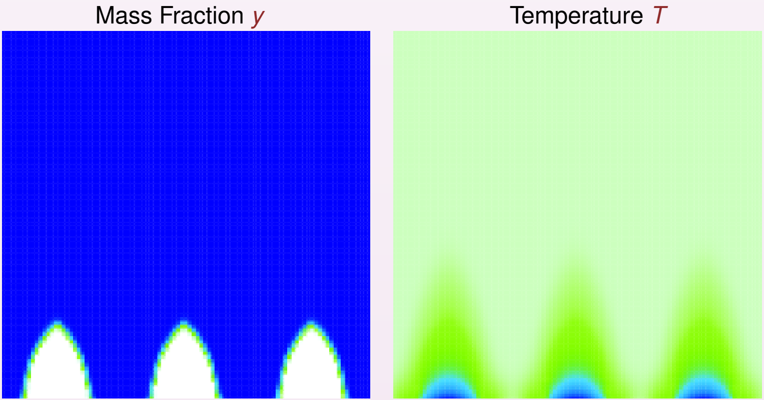

◂ Geometry ▸ Play ▸▸ Skip

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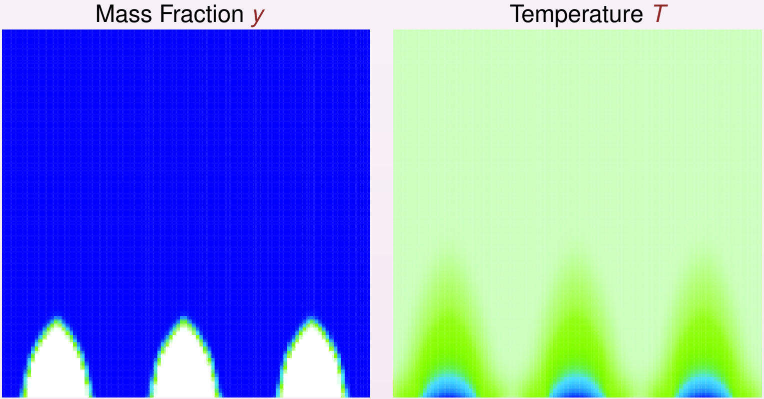
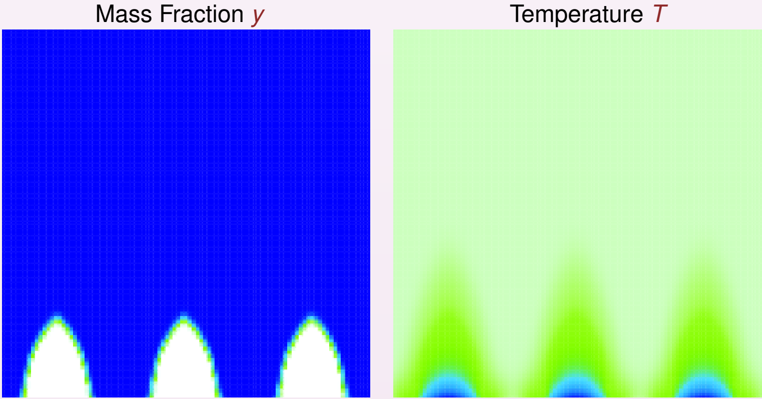


◀ Geometry ▶ Play ⏩ Skip

Nucleating Bubbles



Mass Fraction y

Temperature T

◀ Geometry ▶ Play ⏩ Skip

OUTLINE

1. Model
2. Numerical Method
3. Numerical Tests
4. **Conclusion**

LIQUID-VAPOR PHASE TRANSITION

- Diffuse Interface Model
  - global EOS always at equilibrium (entropy maximization),
  - strict hyperbolicity of the Euler system,
  - uniqueness of Liu solution for the Riemann problem;
- Relaxation Approach
  - 6 (or 5) equation system with relaxation terms;
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  - operator splitting,
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APPENDIX

- Speed of sound
- Isentropic curves
- Surface Tension
- Stiffened Gas for Water
- Tabulated EOS for Water

SPEED OF SOUND

$$c^2 \overset{\text{def}}{=} \tau^2 \left(P^{\mathrm{eq}} \frac{\partial P^{\mathrm{eq}}}{\partial \varepsilon}\bigg|_{\tau} - \frac{\partial P^{\mathrm{eq}}}{\partial \tau}\bigg|_{\varepsilon} \right) = \overset{<0}{\boxed{-\tau^2 T^{\mathrm{eq}}}} \overset{\leq 0}{\boxed{\begin{bmatrix} P^{\mathrm{eq}}, & -1 \end{bmatrix} \begin{bmatrix} s^{\mathrm{eq}}_{\varepsilon\varepsilon} & s^{\mathrm{eq}}_{\tau\varepsilon} \\ s^{\mathrm{eq}}_{\tau\varepsilon} & s^{\mathrm{eq}}_{\tau\tau} \end{bmatrix} \begin{bmatrix} P^{\mathrm{eq}} \\ -1 \end{bmatrix}}}$$

HESSIAN MATRIX OF $\mathbf{w} \mapsto s^{\mathrm{eq}}$

- for all **w** pure phase state

$$\mathbf{v}^T \, \mathrm{d}^2 s^{\mathrm{eq}}(\mathbf{w}) \, \mathbf{v} < 0 \quad \forall\, \mathbf{v} \neq 0,$$

- for all **w** equilibrium mixture state

$$\exists\, \mathbf{v}(\mathbf{w}) \neq 0 \;\text{ s.t. }\; (\mathbf{v}(\mathbf{w}))^T \, \mathrm{d}^2 s^{\mathrm{eq}}(\mathbf{w}) \, \mathbf{v}(\mathbf{w}) = 0.$$

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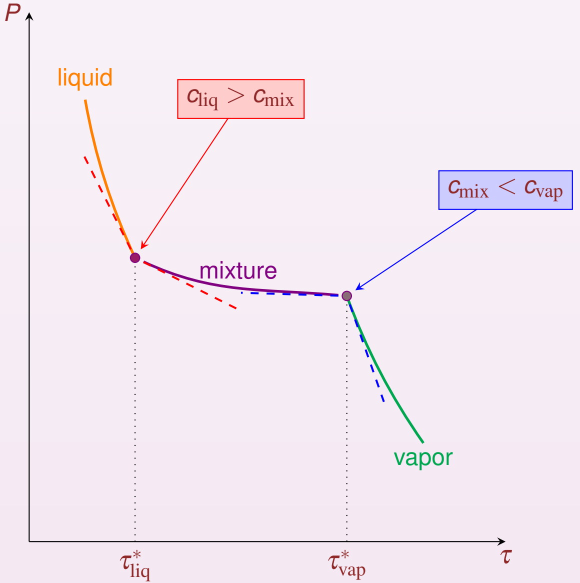
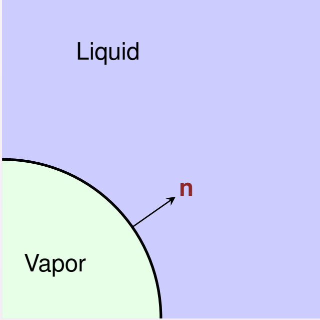
ISENTROPIC CURVES

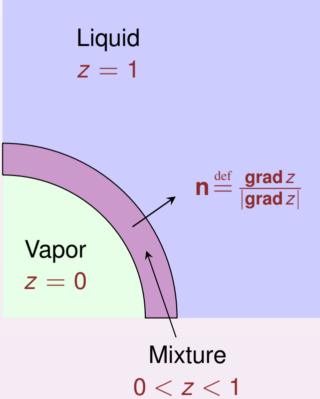


CONTINUUM SURFACE FORCE (CSF) APPROACH

Physical Interface



Diffuse Interface



$$\Pi_{\text{tension}} = -\sigma \operatorname{div}(\mathbf{n})\mathbf{n}$$

[J. U. Brackbill, D. B. Kothe, C. Zemach]

STIFFENED GAS FOR WATER

$$(\tau_\alpha,\varepsilon_\alpha)\mapsto s_\alpha = c_{v_\alpha}\ln(\varepsilon_\alpha - q_\alpha - \pi_\alpha\tau_\alpha) + c_{v_\alpha}(\gamma_\alpha-1)\ln\tau_\alpha + m_\alpha$$

Phase	c_v [J/(kg$\cdot$K)]	γ	π [Pa]	q [J/kg]	m [J/(kg$\cdot$K)]
Water	1816.2	2.35	10^9	-1167.056×10^3	-32765.55596
Steam	1040.14	1.43	0	2030.255×10^3	-33265.65947

TABLE: Parameters proposed by [Le Metayer] for water.

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WATER TABULATED EOS

		Volume (m^3/kg)		Internal Energy (kJ/kg)	
T (K)	P^{sat} (MPa)	τ_{liq}^{sat}	τ_{vap}^{sat}	ε_{liq}^{sat}	ε_{vap}^{sat}
275	0,00069845	0,0010001	181,60	7,7590	2377,5
278	0,00086349	0,0010001	148,48	20,388	2381,6
281	0,0010621	0,0010002	122,01	32,996	2385,7
284	0,0012999	0,0010004	100,74	45,586	2389,8
287	0,0015835	0,0010008	83,560	58,162	2393,9
290	0,0019200	0,0010012	69,625	70,727	2398,0
293	0,0023177	0,0010018	58,267	83,284	2402,1
296	0,0027856	0,0010025	48,966	95,835	2406,2
299	0,0033342	0,0010032	41,318	108,38	2410,3
302	0,0039745	0,0010041	35,002	120,92	2414,4
305	0,0047193	0,0010050	29,764	133,46	2418,4
308	0,0055825	0,0010060	25,403	146	2422,5
311	0,0065792	0,0010071	21,759	158,54	2426,5
314	0,0077262	0,0010082	18,702	171,08	2430,5
317	0,0090418	0,0010094	16,129	183,62	2434,5
320	0,010546	0,0010107	13,954	196,16	2438,5
...	...	...	...	...	...

Source: `http://webbook.nist.gov/chemistry/fluid/`

WATER TABULATED EOS

$$\left.\begin{array}{l} T^i = 278\text{K}\ldots 610\text{K}, \\ \varepsilon_\alpha^{\text{sat}}(T^i),\ \tau_\alpha^{\text{sat}}(T^i) \text{ found in the tables} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathfrak{A} = \left\{\left(T_i, \frac{1}{\varepsilon_{\text{vap}}^{\text{sat}}(T_i)}\right)\right\}_i \\ \mathfrak{B} = \left\{\left(T_i, \frac{\varepsilon_{\text{liq}}^{\text{sat}}(T_i)}{\varepsilon_{\text{vap}}^{\text{sat}}(T_i)}\right)\right\}_i \\ \mathfrak{C} = \left\{\left(T_i, \frac{1}{\tau_{\text{vap}}^{\text{sat}}(T_i)}\right)\right\}_i \\ \mathfrak{D} = \left\{\left(T_i, \frac{\tau_{\text{liq}}^{\text{sat}}(T_i)}{\tau_{\text{vap}}^{\text{sat}}(T_i)}\right)\right\}_i \end{array}\right.$$

$\widehat{\varepsilon}_\alpha^{\text{sat}}$ and $\widehat{\tau}_\alpha^{\text{sat}}$ defined by using a least square approximation of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$:

$$T \mapsto \varepsilon_{\text{vap}}^{\text{sat}} \approx \widehat{\varepsilon}_{\text{vap}}^{\text{sat}} \overset{\text{def}}{=} \frac{1}{\sum_{k=0}^{6} a_k T^k} \qquad T \mapsto \varepsilon_{\text{liq}}^{\text{sat}} \approx \widehat{\varepsilon}_{\text{liq}}^{\text{sat}} \overset{\text{def}}{=} \widehat{\varepsilon}_{\text{vap}}^{\text{sat}}(T) \sum_{k=0}^{6} b_k T^k$$

$$T \mapsto \tau_{\text{vap}}^{\text{sat}} \approx \widehat{\tau}_{\text{vap}}^{\text{sat}} \overset{\text{def}}{=} \frac{1}{\sum_{k=0}^{8} c_k T^k} \qquad T \mapsto \tau_{\text{liq}}^{\text{sat}} \approx \widehat{\tau}_{\text{liq}}^{\text{sat}} \overset{\text{def}}{=} \widehat{\tau}_{\text{vap}}^{\text{sat}}(T) \sum_{k=0}^{9} d_k T^k$$